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EUCLID'S 3 / 35

Euclides

ELEMENTS OF GEOMETRY,

TRANSLATED FROM THE LATIN OF

THE RIGHT REV. THOMAS ELRINGTON, D.D.

LATE LORD BISHOP OF LEIGHLIN AND FERNS, FORMERLY
PROVOST OF TRINITY COLLEGE, DUBLIN.

TO WHICH IS ADDED

A COMPENDIUM OF ALGEBRA, ALSO A COMPENDIUM
OF TRIGONOMETRY.

Designed for the Use of Schools and Private
Instruction.

NEW EDITION, WITH AN APPENDIX.

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PREFACE.

In again presenting this Translation to the Public, it will be only requisite to repeat, that the principal cause of its first appearance was owing to the want of such a publication, in order to combine the study of Euclid with Algebra; and the text of this great *Prelate* being the best adapted for such, induced the Editor to adopt it. We now beg especially to thank the Rev. Doctor Wall, and the Rev. D. Flynn, of Dublin, for the high testimony they have borne to the value of this little volume; and, at the same time, to thank the public at large for its rapid passage through several editions. It would doubtless be of much advantage, even in Classical learning, to commence Mathematical studies at an early age; as the mode of reasoning, and the application of the mental faculties required in learning Euclid and Algebra, expand the mind and give a freedom to thought, which clears the intellect for every other branch of learning. Geometry has been now taught for more than 2000 years, and considered, at all times, the foundation of Science, but its source is uncertain. The history of the rise of Geometry, supported by Diodorus, Strabo, and others, informs us that the Egyptians invented Surveying, for the purpose of preserving in memory the boundaries of their property, yearly destroyed by the overflowing of the Nile. The Jews appear to have been void of any knowledge of Geometry. Some writers tell us, that Pythagoras visited India; this, together with other reasons, induce many to form an opinion, that Geometry came from that country. Pythagoras, who was born about 568 years before the Christian era, was one of the earliest that formed Schools of Geometry; he also discovered the 47th Proposition of the First Book, and is said to have sacrificed 100 head of

cattle for joy. Anaxagoras, born at Clazomene in Ionia, about 500 years before Christ, was an advancer of Science. He taught Philosophy at Athens; Socrates and Pericles, amongst others, were his pupils: but being banished from that city, he retired to the school of his late master at Lampsacus, in which he taught until his death. The magistrates of the town demanded of him how they should honour him after his death; to this he replied that he wished only to be honoured, by the schools of Lampsacus yearly observing the day of his death as a holiday for the boys. The inhabitants erected a tomb to his memory, with the epitaph

> *Ενθαδε πλειστον αληθειας επι τερμα περησας*
> *Ουρανιου κοσμου, κειται Αναξαγορας.*

Plato, who lived 348 years before Christ, was a great advancer of Science. He visited Egypt, and on his return opened a school at Athens, in a grove called the Academy, placing the following inscription over the entrance:—

> *Ουδεις αγεωμετρητος εισιτω.*

There is nothing known of Euclid's birth; however, he had a school at Alexandria, in the time of the first Ptolemy, from which school, amongst others, emanated Archimedes, who was born in Syracuse about 280 years before Christ. Euclid reduced the fundamental principles of Geometry, which had been delivered by Geometricians before him, and added more of his own. Being asked by Ptolemy for an easy mode of acquiring Geometry, he replied that "there was no royal road." He well merited the Elements retaining his name, although it is evident that he was not the author of all under that title; yet even the compilation of such a work would be sufficient to deserve the praise he has received from all enlightened nations. Apollonius was contemporary with Archimedes. The first Latin translation, from the Arabic, appears to have been in the reign of Henry I. by a monk of Bath, named Athelard.

From about the tenth century, the Astronomy, Philosophy, and Physic taught in Europe were principally drawn from Arabian

Schools that were established in Italy and Spain, or from Arabian Sages. Charlemagne, who was crowned Emperor of Rome, A.D. 800, and died A.D. 814, laboured much to cultivate Science. In the eleventh century, the school of Salernum, the chief-town of the Picentini, in Italy, was thought more of than any other, for the study of Physic; yet the medical precepts were drawn from the Saracen schools or the Arabian writers.

In that century, the seven liberal arts were as follows:—GRAMMAR, RHETORIC, LOGIC, ARITHMETIC, MUSIC, GEOMETRY, and ASTRONOMY; the first three were called Trivium, and the schools in which they were taught Triviales; the four last were called Quadrivium; they were also called the four Mathematical arts.

The First Book of Euclid gives the definitions, axioms, and postulates, requisite for establishing the Propositions; it treats of right lines, triangles, &c.

The Forty-eight Propositions here set before us should be well grounded in the memory of every student, before advancing a step beyond them; at the same time becoming acquainted with the Algebra, as set forth in this publication.

The Second Book lays before us the equality between squares and right-angled figures, or squares constructed on the parts of any divided line, &c.

The Third Book treats of the properties of the circle.

The Fourth Book treats of such regular figures as can be described by a circle; and also of the division of the circumference of a circle into equal parts.

The Fifth Book treats of proportion; and the Sixth Book applies it to Geometry, relating to figures which differ only in size.

With paying attention, and not stepping forward too quickly, until all going before is established in the mind, students will advance with pleasure and stability, finding that getting an idea of the utility of Algebra, at the same time, will save them much labour at a more advanced period of their education.

The word Algebra is certainly derived from the Arabic. In

that language the art is called, Al-gjabr W'al-makabala, which is, literally, the Art of Resolution and Equation; therefore, it is probable, that we had the word from the Arabic name of the Art, and not from the Philosopher Geber. It appears that the Arabians received it from the Persians and Indians; but the Persians seem to refer the art to the Greeks: however its source is still disputed.

The portion of Algebra here attached to each book will be found to embrace and simplify all obstructions which usually retard the progress of students. WE trust that the Arithmetical and Algebraic *proofs* of Euclid's Second Book will be found of considerable advantage.

Dr. Isaac Barrow, Tutor to Sir I. Newton, was one of those who first introduced Algebraical symbols into Geometry. The sign $\sqrt{}$ which is derived from the letter *r*, being the initial of radix, or root, was first used to signify the square root, by M. Stifel; the signs $+$ and $-$ were also introduced by him, in the sixteenth century. The sign $=$ was first used to denote equality, by *R. Recorde*, in a Treatise named *The Whetstone of Witte*, published in 1557: and the $\times$ was first used by Oughtrede, in 1631. This Mathematician is said to have died for joy, A.D. 1660, caused by the restoration of King Charles.

It would be indeed advantageous to youth to have this branch taught, combined with Arithmetic and Geometry; therefore, it is the earnest hope of the Editor of this Work, that he may yet see every school adopt (as many have adopted) such a plan of instruction; so that the higher classes, as well as the entrance course of our Universities, may bear the motto of Plato's school at Athens, before quoted: namely—"Let no one ignorant of Geometry enter here." Indeed WE have received the assurance, from many quarters, that pupils have made more advancement when taught according to the system here laid down than in any other way.

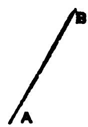
The Compendium of Trigonometry, following the Sixth Book, has been designed, as much as possible, to introduce the Student to the more advanced investigations of that useful Science.

CONTENTS.

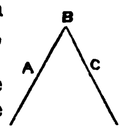
FIRST BOOK

DEFINITIONS.

1. A point is that which is without parts.
2. A line is length without breadth.
3. The extremities of a line are points.
4. A right line, is that which lies evenly between its extremities.



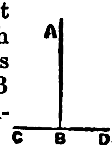
5. A superficies, is that which has only length and breadth.
6. The boundings of a superficies are lines.
7. A plane superficies, is that which lies evenly between its extreme right lines.
8. A rectilineal angle, is the inclination of two right lines to each other, which touch, but do not form one straight line.



An angle is designated either by one letter at the vertex; or three, of which the middle one is at the vertex, the remaining two any place on the legs.

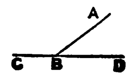
9. The legs of an angle, are the lines which make the angle.
10. The vertex of an angle is the point in which the legs mutually touch each other.

11. When a right line standing on a right line makes the adjacent angles equal, which are ABC and ABD, each of these angles is equal to a right angle, and the right line AB standing on the other, is called the perpendicular.



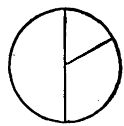
12. The angle ABC, which is greater than a right angle, is called obtuse.

13. The angle A B D, which is less than a right angle, is called acute.



14. A plane figure is a plane superficies, bounded on every side by one or more lines.

15. A circle is a plane figure, contained by one line, which is called the circumference; to which from a certain point within the figure, all right lines drawn are equal.



16. That point is called the centre of the circle.

17. A diameter of a circle is a right line drawn through the centre, and both of its extremities terminate in the circumference.

18. A radius is a right line drawn from the centre to the circumference.

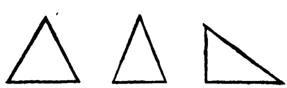
19. A semicircle, is the figure which is contained by the diameter and the part of the circumference which the diameter cuts off.

20. A rectilineal figure, is a plane superficies contained by right lines.

21. A triangle is a plane superficies, which is contained by three right lines.

22. An equilateral triangle is that which has three sides equal.

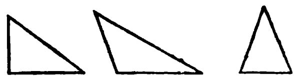
23. An isosceles triangle (or æquicrurum) is that which hath two equal sides.



24. A scalene triangle, is that which hath three unequal sides.

25. A right angled triangle is that which has a right angle.

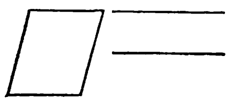
26. An obtuse angled triangleis that which hath an obtuse angle.



27. An acute angled triangle is that which hath three acute angles.

28. Parallel right lines, are those which are in the same plane, and although produced, never meet.

29. A quadrilateral figure, is a rectilineal figure which is contained by four right lines.



30. A parallelogram, is a quadrilateral figure, whose opposite sides are parallel.

31. A square is a quadrilateral figure, which is equilateral and equiangular.

32. Rectilineal figures which have more than four sides, are called polygons.

POSTULATES.

1. Let it be granted that a right line can be drawn from any point to another.

2. That a terminated right line can be produced to any distance.

3. That a circle can be described from any centre, with any radius.

COMMON NOTIONS, OR AXIOMS.

1. Things which are equal to the same are equal to one another.

2. If equals be added to equals, the wholes will be equal.

3. If from equals, equals be taken, the remainders are equal.

4. If to unequals, equals be added, the wholes are unequal.

5. If from unequals, equals be taken, the remainders are unequal.

6. Things which are double of the same, or of equals, are equal to one another.

7. Things which are halves of the same, or of equals, are equal to one another.

8. Magnitudes which coincide with one another are equal to one another.

9. The whole is greater than its part.

10. Two right lines cannot enclose a space:

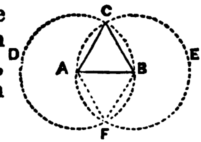
11. All right angles are equal to one another.

12. If two right lines, meeting a right line, make the internal angles on the same side less than two right angles; these two right lines, being produced, will meet one another on that side at which the angles are less than two right angles.

PROPOSITION I. PROBLEM.

On a given finite right line (AB) to describe an equilateral triangle.

From the centre A, and with the interval AB, describe the circle BCD, (by Post. 3). From the centre B and with the interval BA, describe the circle ACE. From the intersection C, draw the right lines CA, and CB, to the extremities of the given right line AB. (by Post. 1).



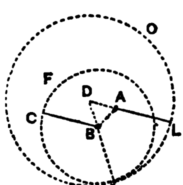
It is manifest that ABC is a triangle, constructed upon the given right line AB; but it is equilateral; for AC is equal to AB, because they are the radii of the same circle, DCB (by Def. 15). But BC is equal to BA, because they are the radii of the same circle ACE. Therefore, since both AC and BC are equal to the same, AB, they are equal to one another (by Ax. 1); and therefore the triangle ACB is equilateral.

SCHOL.—Draw AF and FB, and it can be similarly demonstrated that the triangle AFB is equilateral.

PROPOSITION II. PROBLEM.

From a given point (A) to draw a right line equal to a given finite right line (BC).

From the given point A draw a right line AB to either extremity B of the given right line (by Post. 1); upon AB construct an equilateral triangle ADB (by Prop. 1). From the centre B, with the interval BC, describe a circle GCF (by Post. 3), producing DB, till it meets its circumference in G. From D as a centre, with the interval DG, describe a circle GLO. The circumference of it

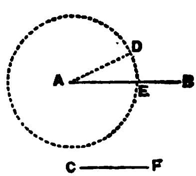


meets DA produced in L. AL is equal to the given right line BC. For DL is equal to DG, because they are the radii of the same circle GLO (by Def. 15); take away both DA and DB, which are also equal (by Const.) and the remainder AL will be equal to the remainder BG (by Ax. 3); but BG is equal to BC, because they are the radii of the same circle GCF: therefore both of them, AL and BC, are equal to the same BG: and therefore are equal to each other (by Ax. 1). Therefore from the point A a right line AL is drawn equal to the given right line BC.

SCHOL.—The position of the right line varies, according to the different extremity of the given line, to which the right line from the given point is drawn, and also according to the different side of that line on which the triangle is constructed.

PROPOSITION III. PROBLEM.

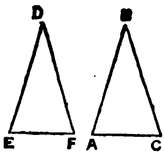
From the greater of two given right lines (AB and CF) to cut off a part equal to the less.



From either extremity A of the greater right line, draw AD, equal to the less CF of the given lines (by Prop. 2). From the centre A, with the interval AD, describe a circle. It cuts off AE equal to AD (by Def. 15), and therefore equal to the given line CF (by Ax. 1).

PROPOSITION IV. THEOREM.

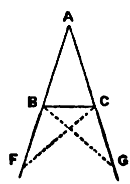
If two triangles (EDF and ABC) have two sides of one equal to two sides of the other (ED and DF to AB and BC); and the angles contained by these sides also equal (D to B); the bases (EF and AC) will be equal; and the angles at the bases, opposite to equal sides, will be equal (E to A and F to C): and also the triangles themselves are equal.



For if the triangles be so applied to each other that the point D may fall on B, and the side DE on BA, and that the sides DF and BC may be at the same side; then, because the sides DE and BA are equal, the point E must fall on A; and, because the angles D and B are equal, the side DF must fall on BC; and, because the side DF is equal to the side BC, the point F must fall on C. But, as the points E and F agree with the points A and C, the right lines FE and AC will agree (by Ax. 10); and therefore the bases EF and AC are equal (by Ax. 8). And, as the legs of the angles E and F agree with the legs of the angles A and C, the angles will agree, and are therefore equal (by Ax. 8). And, as the right lines which contain the triangle EDF agree with the right lines which contain the triangle ABC, the triangles themselves agree, and are therefore equal (by Ax. 8).

PROPOSITION V. THEOREM.

In any isosceles triangle (BAC) the angles at the base (ABC and ACB) are equal: and if the equal sides be produced, the angles below the base (FBC and GCB) will also be equal.



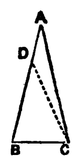
In either of the produced legs, assume any point F, and make AG equal to AF (by Prop. 3); and draw CF and BG. In the triangles FAC and GAB, the sides FA and AC are equal to the sides GA and AB (by Const. and Hypoth.), and the angle A is common, therefore the angle ACF is equal to ABG; and the angle AFC to AGB; and the side FC is equal to the side GB (by Prop. 4). Therefore in the triangles BFC; and CGB, the angle BFC is equal to the angle CGB; and the side CF to the side BG; and taking away the

equals AB and AC from the equals AF and AG, the side BF is also equal to the side CG; therefore the angle FBC is equal to the angle GCB (by Prop. 4); but these are the angles below the base BC. And in the same triangles, the angle FCB is equal to the angle GBC (by Prop. 4); and taking away these from the equals FCA and GBA, the remaining angles ACB and ABC will be equal (by Ax. 3); but these are the angles at the base BC of the given triangle.

COR.—Hence, every equilateral triangle is also equiangular; for whatsoever side is assumed for the base, the adjacent angles will be equal, because the equal sides are opposite.

PROPOSITION VI. THEOREM.

If two angles (B and C) of a triangle (BAC) be equal, the sides opposite to them (AC and AB) are equal.



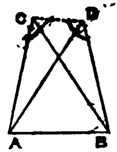
If not, let one of them BA be made greater than the other, cut off a right line BD equal to AC (by Prop. 3), and draw CD. Since, in the triangles DBC, ACB, the sides DB, BC, are equal to the sides AC, CB, and the angles DBC and ACB, which are contained by the equal sides, are also equal (by Hypoth.), the triangles themselves, DBC and ACB, will be equal (by Prop. 4)—a part to the whole, which is absurd: therefore neither of the sides BA nor AC is greater than the other, therefore they are equal.

Hence every equiangular triangle is also equilateral; for whatsoever side is assumed for a base, the angles adjacent to it will be equal, and therefore the sides opposite to them are equal.

PROPOSITION VII. THEOREM.

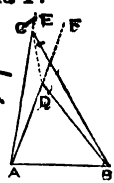
Upon the same right line (AB) and on the same side of it, there cannot be constructed two triangles (ACB, ADB) whose conterminous sides (AC and AD, BC and BD) can be equal.

For, if it be possible, let them be constructed, and first let the vertex of each triangle fall without the other, and draw CD. Because, in the triangle CAD, the sides AD and AC are equal (by Hypoth.), the angles ACD and ADC are equal (by Prop. 5); but ACD is greater than BCD (by Ax. 9), therefore ADC is greater than BCD; but BDC is greater than ADC (by Ax. 9), and therefore BDC is greater than BCD: but in the triangle CBD, the sides BC and BD are equal (by Hypoth.), therefore the angles BDC and BCD are equal (by Prop. 6); but the angle BDC is greater than BCD, which is absurd. Therefore the triangles constructed on the same right line cannot have their conterminous sides equal, if the vertex of each fall without the other.



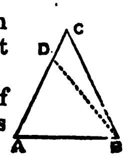
Now let the vertex D of either fall within the other; and draw CD; produce AC and AD to E and F.

Since in the triangle CAD, the sides AC and AD are equal, (by Hypoth.) the angles ECD and FDC are equal, (by Prop. 5); but the angle BDC is greater than FDC (by Ax. 9), therefore also greater than ECD; but ECD is greater than BCD (by Ax. 9), and therefore BDC is greater than BCD; but in the triangle CBD the sides BC and BD are equal (by Hypoth.), therefore the angles B D C BCD are equal (by Prop. 5); but the angle B D f is greater than BCD, which is absurd. Therefore triangles constructed on the same right line, cannot have their conterminous sides equal, if the vertex of one be within the other.



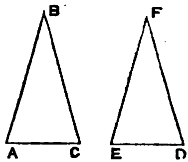
Now let the vertex D of one, fall in the side AC of the other, and it is manifest that the sides AC and AD are not equal.

Therefore, wheresoever the vertices of the triangles fall, the conterminous sides will not be equal.



PROPOSITION VIII. THEOREM.

If two triangles (ABC and EFD) have two sides of one equal to two sides of the other (AB to EF and BC to FD), and also have the base (AC) equal to the base (ED), the angles (B and F) which are contained by the equal sides, will be also equal.

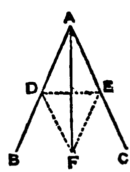


For if the equal bases AC and ED be so applied, that the equal sides AB and EF, CB and DF may be conterminous, the vertex B must fall on F (by Prop. 7); and the equal sides, AB and EF, CB and FD must agree, (by Ax. 10); therefore, the angles B and F must coincide, and therefore are equal (by Ax. 8.)

SCHOL.—It is manifest that the remaining angles A and E, C and D, opposite to the equal sides are equal, and also the triangles themselves.

PROPOSITION IX. PROBLEM.

To cut a given rectilineal angle (BAC) into two equal parts.



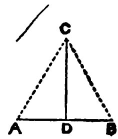
From A take the equals AD and AE (by Prop. 3), draw DE, and upon it construct an equilateral triangle DFE (by Prop. 1). The right line joining the points A and F will bisect the given angle BAC.

For in the triangles FAD, FAE, the sides AD and AE are equal (by Constr.), and AF common, and the base DF is equal to the base EF (by Constr.) therefore the angles DAF and EAF are equal (by Prop. 8); and therefore the right line AF bisects the given angle BAC.

COR.—By this proposition an angle can be also divided into 4, 8, 16, parts, &c. by again bisecting each part.

PROPOSITION X. PROBLEM.

To cut a given right line (AB) into two equal parts.

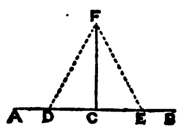


Upon the given right line construct an equilateral triangle ACB (by Prop. 1); bisect the angle ACB by the right line CD (by Prop. 9); this will bisect the given right line in the point D.

For in the triangles ACD, BCD, the sides AC and CB are equal (by Constr.), but CD is common, and the angles ACD, BCD are equal (by Constr.), therefore the bases AD and DB are equal (by Prop. 4), and therefore the given right line AB is bisected in D.

PROPOSITION XI. PROBLEM.

To draw a perpendicular to a given right line (AB), from a point (C) in the given line.



From the given point C make both CD and CE equal (by Prop. 3); upon DE construct an equilateral triangle DFE; draw FC, and it will be perpendicular to the given line.

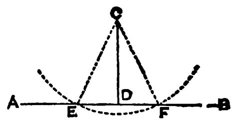
For in the triangles DFC, EFC, the sides DF and DC are equal to the sides EF and EC (by Constr.) and CF is common; therefore the angles DCF and ECF, opposite to the equal sides DF and EF are equal (by Schol. Prop. 8); and therefore FC is perpendicular to the given right line AB (by Def. 11.)

SCHOL.—By the same method a perpendicular can be erected at the extremities of a given right line, if the right line be first produced.

PROPOSITION XII. PROBLEM.

From a given point (C) without a given indefinite right line (AB) to draw a perpendicular to it.

From the centre C describe a circle, cutting the given right line in E and F. Bisect EF as in D (by Prop. 10), and from the given point draw CD to the point of bisection; it will be perpendicular to the given line.

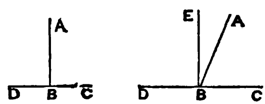


For draw CE and CF; and in the triangles EDC and FDC, the sides EC, FC will be equal (by Def. 15), but ED, FD are also equal (by Constr.) and CD common; therefore the angles EDC and FDC, opposite to the equal sides EC and FC, are equal (by Schol. Prop. 8); and therefore DC is perpendicular to the given right line AB (by Def. 11.)

PROPOSITION XIII. THEOREM.

When a right line (AB) standing upon a right line (DC), makes angles with it, they shall be either two right angles, or equal to two right angles.

If a right line AB, be perpendicular to the right line DC, the angles ABC and ABD are right angles (by Def. 11). But if not, draw BE perpendicular to DC, and it is manifest that the angles CBA and ABD are equal to the angles CBE and EBD, and therefore equal to two right angles.



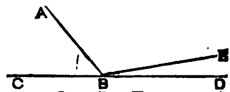
Cor. 1.—If several right lines stand on the same right line at the same point, they will make angles equal to two right angles.

Cor. 2.—Two right lines intersecting will make angles equal to four right angles.

Cor. 3.—If many right lines intersect one another in the same point, all the angles taken together are equal to four right angles.

PROPOSITION XIV. THEOREM.

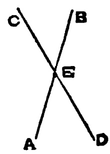
If two right lines (CB and BD) meeting another right line (AB) at the same point, and at different sides, make angles with it, which are equal to two right angles; these right lines (CB and BD) will form one straight line.



For if not, if it be possible, let BE be in the same right line with CB, and the angles CBA and ABE will be equal to two right angles (by Prop. 13), but CBA and ABD are equal to two right angles (by Hypoth.); therefore CBA and ABE are equal to CBA and ABD: take away the common part CBA, and ABE will be equal to ABD; a part to the whole, which is absurd. Therefore BE is not in the direct line with CB; and it can be similarly proved that no other line except BD is in the direction of it, therefore BD is a continuation of BC.

PROPOSITION XV. THEOREM.

If two right lines (AB and CD) cut one another, the vertical angles will be equal (CEA to BED and CEB to AED).

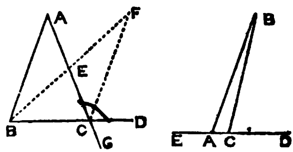


For since the right line CE stands on the right line AB, the angle AEC together with the angle CEB is equal to two right angles (by Prop. 13); and because the right line BE stands on CD, the angle CEB together with BED is equal to two right angles (by Prop. 13); therefore AEC together with CEB are equal to CEB together with BED (by Ax. 1); take away the common part CEB, and AEC will be equal to BED. It can be similarly demonstrated that CEB and AED are equal.

PROPOSITION XVI. THEOREM.

If any side (BC) of a triangle be produced, the external angle (ACD) will be greater than either of the internal remote angles (A or B).

Bisect the side AC in E (by Prop. 10), and draw BE, produce it so that EF may be equal to BE (by Prop. 3), and draw CF.



In the triangles CEF and AEB, the sides CE and EF are equal to the sides AE and EB (by Constr.); and the angle CEF is equal to the angle AEB (by Prop. 15), therefore the angles ECF and A are equal (by Prop. 4); and therefore ACD is greater than A. It can be similarly proved, by producing AC, that the angle BCG is greater than the angle B, and therefore that ACD, which is equal to BCG (by 15 Prop.) is greater than B.

Cor. 1.—If from the point B, two right lines be drawn to the right line ED, one of them BA perpendicular to it, and the other BC not, the perpendicular falls at the side of the acute angle.

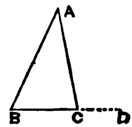
For, if it be possible, let BA, perpendicular to the right line ED, fall at the side of the obtuse angle BCE, and the angle BAE will be less than the angle BCE (by Def. 12); but BAE is greater than BCE (by Prop. 16), which is absurd: therefore BA does not fall at the side of the obtuse angle, therefore it falls at the side of the acute angle.

Cor. 2.—Two perpendiculars cannot be drawn from the same point B, to the same right line ED.

For, if it be possible, let BA and BC be both perpendicular to the right line ED, and the angle BAE will be equal to the angle BCE (by Ax. 11), but it is also greater (by Prop. 16), which is absurd: therefore the right lines BA and BC are not both perpendicular to ED

PROPOSITION XVII. THEOREM.

Any two angles of a triangle (BAC) are together less than two right angles.

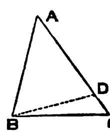


For, produce any side BC, and the angle ACD will be greater than either of the angles A or B (by Prop. 16); therefore ACB together with either A or B is less than ACB together with ACD, and therefore less than two right angles (by Prop. 13.) It can be similarly shown, by producing CB from the angle B, that the angle ABC together with A is less than two right angles. Therefore any two angles of a triangle are less than two right angles.

COR.—If in any triangle, one angle be obtuse or right, the remaining angles will be acute, and if two angles be equal, they are acute.

PROPOSITION XVIII. THEOREM.

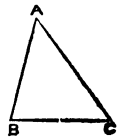
If in any triangle (BAC), one side (AC) be greater than the other (AB), the angle opposite to the greater side will be greater than the angle which is opposite to the less side.



From the greater side AC, cut off AD equal to the less side (by Prop. 3) and conterminous with it, and join BD. Because the triangle BAD is isosceles (by Constr.) the angles ABD and ADB are equal (by Prop. 5); but ADB is greater than the internal angle ACB (by Prop. 16), therefore ABD is greater than ACB, and therefore ABC is greater than ACB. But ABC is opposite the greater side AC, and ACB is opposite the less side AB.

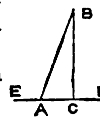
PROPOSITION XIX. THEOREM.

If in any triangle (BAC) one angle (ABC) be greater than the other (C), the side (AC) opposite the greater angle, will be greater than the side (AB) which is opposite the less angle.



For the side AC is either equal, or less, or greater, than the side AB. It is not equal to AB, for if it be, the angles B and C would be equal (by Prop. 5), which is contrary to the Hypothesis.

It is not less than AB; for if it be, the angle B would be less than C (by Prop. 18), which is contrary to Hypothesis. Since therefore the side AC is neither equal to, nor less than the side AB, it is greater than it.

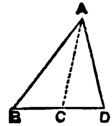


Cor.—If from the same point B, to the same right line ED, two right lines BC and BA be drawn, the one BC perpendicular, and the other not, BC will be less than BA.

For in the triangle ABC, since the angle BCA is right, BAC will be acute (by Cor. Prop. 17), therefore BC is opposite the less angle, and therefore is less (by Prop. 19) than BA, which is opposite the greater.

PROPOSITION XX. THEOREM.

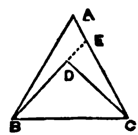
In any triangle (BAD) any two sides together (BA and AD) are greater than the remaining side (BD).



Let the angle BAD be bisected by the right line AC; and then because the external angle BCA is greater than the internal angle CAD (by Prop. 16), but BAC is equal to CAD (by Constr.), BCA is greater than BAC, and therefore the side BA is greater than BC (by Prop. 19); and in the same manner the side AD is greater than DC; therefore the two sides BA, AD, are greater than the two sides BC, CD, or the side BD. And in like manner by bisecting any angle, it may be demonstrated that the sides about it are greater than the remaining side.

PROPOSITION XXI. THEOREM.

If from any point (D) within a triangle (BAC) there be drawn two right lines, (DB and DC) to the extremities of any side, (BC); these are less than the two remaining sides of the triangle, but contain a greater angle.



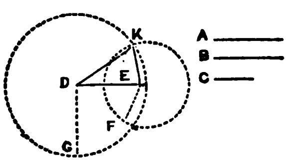
Produce BD to E; and because in the triangle BAE, the sides BA and AE, are greater than the remaining side BE, (by Prop. 20), add EC to both, and the sides BA and AC, will be greater than BE and EC; but in the triangle EDC, the two sides DE and EC are greater than the remaining side DC, (by Prop. 20,) therefore if BD be added to both, BE and EC will be greater than BD and DC, but BA and CA are greater than BE and EC; therefore BA and AC, are greater than BD and DC.

Because the external angle BDC is greater than the internal angle DEC, (by Prop. 16,) and DEC is similarly greater than BAE, (by Prop. 16,) the angle BDC will be greater than BAE.

SCHOL.—The excess of the angle BDC above BAE is equal to the two angles ABD and ACD, as appears from Prop. 32.

PROPOSITION XXII. PROBLEM.

Given three right lines (A, B, and C), of which any two together are greater than the remaining line, to construct a triangle, whose sides will be equal to the given lines.



To any point D, draw the right line DE, equal to the given line A, (by Prop. 2,) and to the same point draw DG, equal to the given line B (by Prop. 2), and to the point E draw EF equal to the given line

C

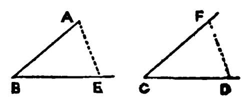
C (by Prop. 2.) From the centre D, with the interval DG, describe a circle, and from the centre E, with the interval EF, describe a circle (by Post. 3); from the point of intersection K, draw KD and KE.

It appears that the sides DE, DK and KE of the triangle DKE are equal to the given lines A, B and C.

Cor.—Hence a triangle can be constructed equal to a given one, namely, by constructing a triangle whose sides are equal to the given sides; for this will be equal to the given triangle (by Schol. Prop. 8).

PROPOSITION XXIII. PROBLEM.

On a given right line (BE) and at a given point in it (B) to construct an angle equal to a given angle (C).

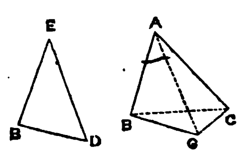


Draw any right line FD, cutting the legs of the given angle C, and construct EBA equilateral to the triangle DCF, whose sides AB and EB meeting at the given point B, may be equal to the legs of FC and DC of the given angle C (by Prop. 22.) The angle B will be equal to the given angle C.

For since the triangles DCF and EBA are equilateral, the angles C and B opposite the equal sides DF and EA are equal (by Prop. 8).

PROPOSITION XXIV. THEOREM.

If two triangles (BED, BAC) have two sides of the one equal to two sides of the other (BE to AB and ED to AC), and if one of these angles (BAC) be greater than the other (E), the side (BC) opposite the greater, will be greater than (BD) opposite the less angle.



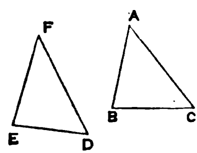
At the point A and with the side AB which is not the greater, construct an angle BAG equal to BED (by Prop. 23), and make AG equal to ED (by Prop. 3), and draw BG and GC

Because the right lines AG and AC are equal (by Hypoth. and Constr.) the angles ACG and AGC will be equal (by Prop. 5): but BGC is greater than AGC, and therefore greater than ACG, and also greater than BCG.

Then in the triangle BGC, the angle BGC is greater than BCG, and therefore the side BC is greater than the side BG (by Prop. 19); but BG is equal to BD (by Constr. and Prop. 4), and therefore BC is greater than BD.

PROPOSITION XXV. THEOREM.

If two triangles (BAC and EFD) have two sides of the one equal to two sides of the other (BA to EF and AC to FD), but the remaining side (BC) greater than the remaining side (ED), the angle A opposite the greater side will be greater than the angle (F) opposite the less.



The angle A is either equal to F, or less than it, or greater than it.

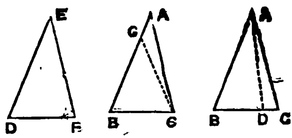
It is not equal to it; for if it were, the side BC should be equal to the side ED (by Prop. 4), contrary to the Hypothesis.

It is not less; for if it were, the side BC should be less than ED (by Prop. 19), contrary to Hypothesis.

Therefore because the angle A is neither equal to the angle F, nor less than it, it will be greater.

PROPOSITION XXVI. THEOREM.

If two triangles (BAC, DEF) have two angles of the one equal to two angles of the other (B to D and C to F); and a side of one equal to a side of the other, that is, either the sides which are between the equal angles (as BC to DF) or opposite to the equal angles (as BA to DE), the remaining sides and angle of the one are equal to the remaining angle and sides of the other.



First, let the side BC be equal to the side DF, then the side BA will be also equal to the side DE.

For they are not unequal, but if it be possible to be done, let one of them BA be greater than the other, and cut off the right line BG equal to the less DE, and draw CG. In the triangles GBC, EDF, the sides GB, BC are equal to the sides ED, DF (by Constr. and Hypoth.), and the angle B is equal to the angle D (by Hypoth.), therefore the angles GCB and F are equal (by Prop. 4); but the angle BCA is equal to the angle F (by Hypothesis), therefore BCG is equal to BCA (by Ax. 1), which is absurd; therefore neither of the sides BA and DE is greater than the other; BA and DE are therefore equal; but BC and DF are equal (by Hypoth.), and the angles B and D are equal (by Hypoth.), therefore the remaining side AC is equal to the remaining side EF, and also the remaining angle A to the remaining angle E (by Prop. 4).

Again, let the equal sides BA and DE be taken, which are opposite to the equal angles C and F; and the sides BC and DF will be also equal.

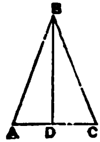
For they are not unequal, but if it be possible to be done, let one of them BC be greater than the other, and cut off the right line BD equal to the less DF, and draw AD.

In the triangles ABD, EDF, the sides AB, BD are equal to the sides ED, DF (by Constr. and Hypoth.) and the angle B is equal to the angle D (by Hypoth.); therefore the angles ADB and F are equal (by Prop. 4), but the angle C is equal to the angle F (by Hypoth.) ADB is therefore equal to C (by Ax. 1); which cannot be possible (by Prop. 16); therefore neither of the sides BC and DF is greater than the other; BC and DF are therefore equal; BA and DE are equal (by Hypoth.)

and the angles B and D are equal (by Hypoth.), therefore the remaining side AC is equal to the remaining side EF, and the remaining angle A to the remaining angle E (by Prop. 4).

SCHOL.—It appears that the triangles themselves are also equal.

COR. 1.—In an isosceles triangle ABC, a right line BD drawn from the vertex perpendicular to the base, bisects both the base and the vertical angle.



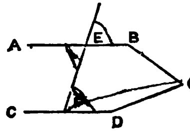
For in the triangles ABD, CBD, the angles A and ADB are equal to the angles C and CDB (by Hypoth.), but the side BD, opposite to the equal angles A and C, is common, therefore the angles ABD, CBD are equal, and also the sides AD and DC (by Prop. 26); the vertical angle and the base are therefore bisected.

COR. 2.—It appears from Proposition 4th of this Book, that the right line bisecting the vertical angle of an isosceles triangle, also bisects the base, and is perpendicular to it; and from Prop. 8, that the right line drawn from the vertical angle bisecting the base, is perpendicular to it, and bisects the vertical angle.

PROPOSITION XXVII. THEOREM.

If a right line (EF) cutting two right lines (AB and CD), make the alternate angles equal (AEF to EFD) these right lines will be parallel.

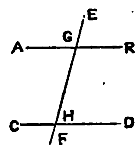
For suppose them not to be parallel, but meeting, if it be possible, in G, and the external angle AEF of the triangle EGF is greater than the internal angle EFG (by Prop. 16), but it is equal (by Hypoth.), which is absurd; therefore the right lines AB and CD do not meet towards B, D. It can be



similarly demonstrated, that they do not meet at the side A, C. Therefore, since they meet at neither side, they are parallel (by Def. 28).

PROPOSITION XXVIII. THEOREM.

If a right line (EF) cutting two right lines (AR and CD), make the external angle equal to the internal opposite angle on the same side of the line EF (EGA to GHC, or EGR to GHD); or make the internal angles at the same side of the line (AGH and CHG, or RGH and DHG) equal to two right angles, these right lines will be parallel.



First, let EGA and GHC be equal; and because EGA is equal to RGH (by Prop. 15), GHC and RGH will be equal, and are alternate angles, the right lines AR and CD are therefore parallel (by Prop. 27).

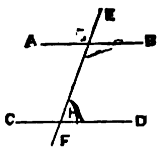
It may be similarly demonstrated, if the angles EGR and GHD be equal.

Now let AGH and CHG together be equal to two right angles; and because GHD and CHG are also equal to two right angles (by Prop. 13), AGH and CHG together will be equal to GHD and CHG taken together (by Ax. 1); take away the common angle CHG and AGH will be equal to GHD, but they are alternate angles, therefore the lines AR and CD are parallel (by Prop. 27).

It can be similarly demonstrated, if RGH and DHG be equal to two right angles.

PROPOSITION XXIX. THEOREM.

A right line (EF) cutting parallel right lines, (AB and CD) makes the alternate angles equal (AGH to GHD, and CHG to HGB); and the external angle equal to the internal opposite angle (EGA to GHC, and EGB to GHD); and also the internal angles at the same side of the right line (AGH and CHG, BGH and DHG) equal to two right angles.



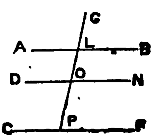
First, the alternate angles AGH and GHD are equal. For if not, let one of them, AGH, be greater than the other; and by adding BGH to both, AGH and BGH together will be greater than BGH and GHD; but AGH and BGH are equal to two right angles (by Prop. 13), therefore BGH and GHD are less than two righ angles; and therefore the right lines AB and CD would meet at the side B,D (by Ax. 12), but they are parallel (by Hypoth.), and therefore cannot meet, which is absurd; therefore neither of the angles AGH and GHD is greater than the other; they are therefore equal. It can be similarly demonstrated, that BGH and GHC are equal.

Secondly, the external angle EGB is equal to the internal GHD. For EGB is equal to the angle AGH (by Prop. 15), and AGH is equal to the internal angle GHD (by part 1st), therefore EGB is equal to GHD. It can be similarly demonstrated, that EGA and GHC are equal.

Thirdly, the internal angles at the same side, BGH and GHD, are equal to two right angles. For since the alternate angles GHD and AGH are equal (by part 1st), by adding BGH to both, BGH and GHD will be equal to BGH and AGH, and therefore equal to two right angles (by Prop. 13). It can be similarly demonstrated, that the angles AGH and GHC are equal to two right angles.

PROPOSITION XXX. THEOREM.

If two right lines (AB, CF) be parallel to the same right line (DN), they will be parallel to one another.

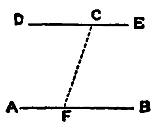


For let the right line GP cut them. The external angle GLB is equal to the internal LON (by Prop. 29), and the angle LON is similarly equal to the angle OPF, therefore GLB is equal to

OPF (by Ax. 1), and the right lines AB and CF are therefore parallel (by Prop. 28).

PROPOSITION XXXI. PROBLEM.

To draw a right line parallel to a given right line, (AB) through a given point (C) without it.

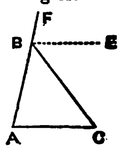


Draw the right line CF, cutting the given line AB. At the point C, and with the right line CF, construct an angle FCE, equal to the angle AFC (by Prop. 23), but at the opposite side of the right line CF; DE will be parallel to the given right line AB.

For the right line CF, cutting the lines DE and AB, makes the alternate angles ECF and AFC equal, and therefore these right lines are parallel (by Prop. 27.)

PROPOSITION XXXII. THEOREM.

In every triangle, if any side (AB) be produced, the external angle (FBC) is equal to the two internal remote angles (A and C). And the three internal angles are equal to two right angles.



Through B draw BE parallel to AC (by Prop. 31). The angle FBE is equal to the internal remote angle A (by Prop. 29), and the angle EBC is equal to the alternate angle C (by Prop. 29), therefore the whole angle FBC is equal to the two angles A and C.

The angle ABC with FBC is equal to two right angles (by Prop. 13), but FBC is equal to the two angles A and C (by part 1st), therefore the angle ABC with the two angles A and C, is equal to two right angles.

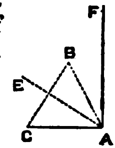
Cor. 1.—If in a triangle, one angle be right, the remaining two together are equal to a right angle.

Cor. 2.—If in two triangles, there be two angles in each respectively equal, the remaining angles will be also equal.

COR. 3.—In an isosceles right angled triangle, each angle at the base is half a right angle.

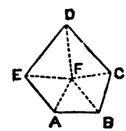
COR. 4.—In an equilateral triangle, each angle is a third part of two right angles.

COR. 5.—Hence it is possible to trisect a right angle FAC. For assume any part of the leg AC, and construct an equilateral triangle CBA, whose angle CAB bisect with the right line AE; and because CAB is a third part of two right angles (by Cor. 4), it will be equal to two-thirds of one right angle, and therefore BAF is a third part of a right angle, and BAE, EAC, and FAB are therefore equal.

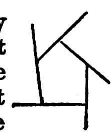


COR. 6.—All the angles together of any rectilineal figure ABCDE, are equal to twice as many right angles, deducting four, as the figure hath sides.

For assume a point F within the figure, and draw the right lines FA, FB, FC, FD and FE. There are as many triangles constructed as the figure has sides, and therefore all these angles will be equal to twice as many right angles as the figure has sides (by Prop. 32); from these take four right angles, for the angles at the point F (by Cor. 3, Prop. 13), and the remainder, namely, the angles of the figure, will be equal to twice as many right angles, deducting four, as there are sides of the figure.

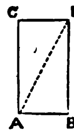


COR. 7.—All the external angles of any rectilineal figure are equal to four right angles. For each external angle with the internal adjacent, are equal to two right angles (by Prop. 13); therefore all the external with all the internal, are equal to twice as many right angles as there are sides of the figure; but the internal with four right angles, are equal to twice as many right angles as there are sides of the figure (by Cor. 6); take away from both the internal angles, and the external angles will be equal to four right angles.



PROPOSITION XXXIII. THEOREM.

Right lines (AC and BD) joining the adjacent extremities of equal and parallel right lines (AB and CD) are themselves equal and parallel.



For draw the diagonal AD, and in the triangles CDA, BAD, the sides CD and BA are equal (by Hypoth.), but AD is common, and the angle CDA is equal to the alternate angle BAD (by Prop. 29), therefore the right lines AC and BD are equal (by Prop. 4); therefore the right line AD, cutting the right lines AC and BD, makes the alternate angles equal, and AC and BD are therefore parallel (by Prop. 27).

PROPOSITION XXXIV. THEOREM.

The opposite sides (AB and CD, AC and BD) of a parallelogram (AD) are equal, and the opposite angles (A and D, C and B) are also equal, and it is bisected by a diagonal.

For in the triangles CDA, BAD, the angles CDA and BAD, CAD and BDA are equal, for they are the alternate angles (by Prop. 29), and the side AD between the equal angles is common to both triangles, therefore the sides CD and CA are equal to AB and BD (by Prop. 26), and the triangle CDA is equal to the triangle BAD (by Sch. Prop. 26), and the angles ACD and ABD are also equal (by Prop. 26); and because ACD together with CAB is equal to two right angles (by Prop. 29), and ABD with CDB is also equal to two right angles, if the equals ACD and ABD be taken from both, the remaining angles CAB and CDB will be also equal.

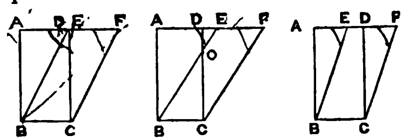
Cor. 1.—In any parallelogram if one angle be right, the remaining angles will be right angles.

For the adjacent angle is right, because with a right angle it is equal to two right angles (by Prop. 29); and the opposite angles are right, since they are equal to these right angles (by Prop. 34).

COR. 2.—If two parallelograms have an angle of the one equal to an angle of the other, the remaining angles will be also equal; for the angles which are opposite to these equal angles are equal to them (by Prop. 34), and therefore are equal to one another; but the adjacent angles are also equal to one another, because together with these equals, they are equal to two right angles (by Prop. 29).

PROPOSITION XXXV. THEOREM.

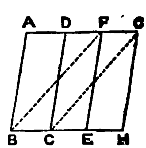
Parallelograms (BD and BF) on the same base and between the same parallels are equal.



Because the right lines AD and EF are equal to the same right line BC (by Prop. 34), they are equal to each other, therefore, by adding DE to both, as in figure 2, and by subtracting DE from both, as in figure 3, the right lines AE and DF are equal; also the right lines BA and BE are equal to the right lines CD and CF (Prop. 34), therefore the triangles BEA and CFD are equal (by Schol. Prop. 8); but subtract BEA from the quadrilateral BAFC, the remainder is the parallelogram BF; and subtract the triangle CFD from the same quadrilateral, the remainder is the parallelogram BD; therefore the parallelograms BD and BF are equal (by Ax. 3).

PROPOSITION XXXVI. THEOREM.

Parallelograms (BD and EG) on equal bases, and between the same parallels are equal.

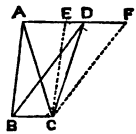


For draw BF and CG, and because BC and FG are equal to the same EH (by Hypoth. and Prop. 34), they are equal to each other, but they are also parallel (by Hypoth.), therefore BF and CG joining them, are parallel (by

Prop. 33), and BG is a parallelogram, therefore it is equal to both BD and EG (by Prop. 35), and therefore BD and EG are equal to one another (by Ax. 1).

PROPOSITION XXXVII. THEOREM.

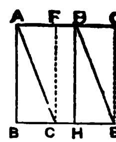
Triangles (BAC and BDC) on the same base, and between the same parallels, are equal.



For draw through the point C, the right lines CF and CE, parallel to BA and BD (by Prop. 31), and the parallelograms BAEC, BDFC will be equal (by Prop. 35), but the triangles BAC and BDC are halves of them (by Prop. 34), and therefore are also equal (by Ax. 7).

PROPOSITION XXXVIII. THEOREM.

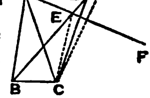
les (BAC and HDE) on equal bases, and between the same parallels, are equal.



For draw through the points C and E the right lines CF and EG, parallel to the right lines BA and HD (by Prop. 31); and the parallelograms BAFC, HDGE, will be equal (by Prop. 36), but the triangles BAC and HDE are halves of them, (by Prop. 34), and therefore are also equal (by Ax. 7).

PROPOSITION XXXIX. THEOREM.

Equal triangles (BAC and BDC) on the same base, and on the same side of it, are between the same parallels.



For if AD be not parallel to BC, draw through the point A the right line AF parallel to BC, cutting the side BD of the triangle BDC in the point E, different from the vertex, and draw CE.

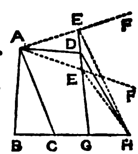
Because AF and BC are parallel, the triangle BEC is equal to the triangle BAC (by Prop. 37), but BDC

is equal to BAC (by Hypoth.), therefore BEC and BDC are equal, a part and the whole, which is absurd (by Ax. 1). Therefore AF is not parallel to BC, and it may be similarly shown that no other line except AD is parallel to it, therefore AD is parallel to BC.

PROPOSITION XL. THEOREM.

Equal triangles (BAC and GDH) on equal bases, and at the same side of it, are between the same parallels.

For if AD be not parallel to BH draw through A the right line AF, p rallel to BH, cutting one of the sides GD of the triangle GDH, in the point E, different from the vertex, and draw HE.

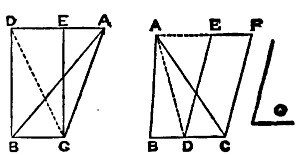


Because AF is parallel to BH, and BC and GH are equal, the triangle GEH is equal to the triangle BAC (by Prop. 38), but GDH is equal to BAC (by Hypoth.), therefore GEH and GDH are equal, a part and the whole equal to each other (by Ax. 1), which is absurd. Therefore AF is not parallel to BH, and it can be similarly proved that no other is parallel except AD, therefore AD is parallel to BH.

PROPOSITION XLI. THEOREM.

If a parallelogram ([illegible]) and a triangle (BAC) have the same base, and be between the same parallels, it will be double of the triangle.

For draw CD. The triangle BDC is equal to the triangle BAC (by Prop. 37), but BE is double of the triangle BDC (by Prop 34), therefore BE is also double of BAC.



SCHOL.—Hence it appears that a parallelogram is double of a triangle, if they have equal bases, and be between the same parallels.

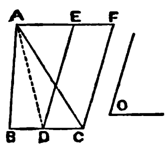
COR.—If a triangle BAC and a parallelogram DF be between the same parallels, but the base BC of the triangle be double of the base DC of the parallelogram the triangle and the parallelogram will be equal.

For draw AD, and since the triangles BAD, DAC are equal (by Prop. 38), BAC is double of the triangle DAC, but the parallelogram DF, is double of DAC (by Prop. 41), therefore the triangle BAC and the parallelogram DF are equal (by Ax. 6).

PROPOSITION XLII. PROBLEM.

To construct a parallelogram equal to a given triangle (BAC) having an angle equal to a given angle (O).

Through A draw AF parallel to BC; bisect the base BC of the triangle, as in D, and at the point D, and with the right line CD, construct an angle CDE, equal to the given angle O; through C draw CF parallel to DE, meeting the right line AF, in F; DF will be the parallelogram sought for.

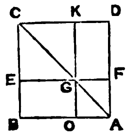


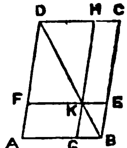
For since EF is parallel to DC, (by Construction,) and CF to DE, (by Construction,) DEFC is a parallelogram, by Def. 30); it has likewise an angle CDE, equal to the given one O (by Const.). But it is equal to the triangle BAC, because it is between the same parallels, and has for a base, half the base of the triangle (by Cor. Prop. 41).

PROPOSITION XLIII. THEOREM.

In a parallelogram, (AC,) the complements (AK and KC) of the parallelograms (FH and GE), which are about the diagonal, are equal.

Draw the diagonal DB, through any point in it, as K, draw the right lines FE & GH, parallel to AB & BC, and FH and GE will be the parallelograms about the diagonal; and AK and KC the complements of them.





For since the triangles BAD, BCD are equal (by Prop. 34), and the triangles BGK, KFD are likewise equal to BEK, KHD, (by Prop. 34,) by taking away these equals BGK and KFD, BEK and KHD, from the equals BAD and BCD, the remainders, namely, the parallelograms AK and KC, will be equal.

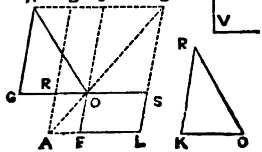
Cor.—Parallelograms OF and EK, about the diagonal of a square, AC, are squares.

For since the triangle BAC is isosceles, but the angle at B right, the angle BAC will be half a right angle (by Cor. 3, Prop. 32); therefore, since in the triangle OAG, the angle at O is right, for it is external to the right angle ABC, (Prop. 29,) and the angle OAG half a right angle, OGA will be likewise half a right angle; and therefore OG and OA are equal (by Prop. 6); therefore, it is manifest that OF is a square. And it can be similarly demonstrated that EK is a square.

PROPOSITION XLIV. PROBLEM.

To a given right line (OS) to apply a parallelogram, equal to a given triangle, and having an angle equal to a given angle (V).

First, let the given triangle be GHO, one of whose sides GO, forms a continuation of the given right line OS. Bisect GO in R, and upon RO construct a parallelogram RC equal to the given triangle, and having an angle BRO equal to the given angle



V (by Prop. 42); through S draw SD parallel to either OC or RB, so that it may meet BC produced, in D; and draw DO that it may meet BR produced in A; through A draw AL parallel to either RS or BD, and produce CO and DS to F and L.

The parallelogram FS, equal to the given triangle GHO, will be thus applied to the given line OS, and have an angle equal to the given angle V.

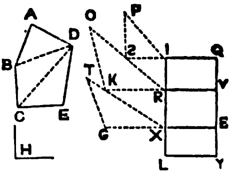
Because ABDL is a parallelogram (by Constr.), and FS and RC are the complements of the parallelograms about the diagonal, FS will be equal to RC (by Prop. 43); but RC and the given triangle are equal (by Constr.); therefore FS is equal to the triangle GHO (by Ax. 1): and because the angle OFL is equal to the internal angle BAF (by Prop. 29), and BAF equal to the external angle BRO (by Prop. 29), OFL and BRO are equal; but BRO and the given angle V are also equal (by Constr.), therefore OFL and the given angle V are equal.

But next, if there be given a triangle KRO, none of whose sides forms a direct right line with the given line OS, then produce the given line OS, and make the produced part OG, equal to any of the sides of the given triangle as KO, and on it construct a triangle GHO equal to KRO (by Cor. Prop. 22), and apply to the given line OS a parallelogram equal to GHO (by the first part of this Prop.); it will be likewise equal to the triangle KRO, as is manifest.

PROPOSITION XLV. PROBLEM.

To construct a parallelogram equal to a given rectilineal figure (ABCED), having an angle equal to a given angle (H).

Let the given rectilineal figure be resolved into triangles. Construct IPS equal to the triangle BDA (by Cor. Prop. 22), and construct a parallelogram RQ equal



to IPS (by Prop. 44), having an angle RIQ equal to the given one H; produce VR, and make the produced part RK equal to the side of either triangle BDC, as BD, and on it construct a triangle RKO, equal to the triangle DBC (by Cor. Prop. 22), and construct a parallelogram XV equal to it, on RV, and having an angle XRV equal to one of the angles RIQ or H (by Prop. 44); and thus construct parallelograms equal to the remainder of all the triangles, into which the given figure is resolved; the parallelogram ILYQ will be equal to the given rectilineal figure ABCED, and having an angle LIQ equal to the given angle H.

For since RV and IQ are parallel, the angle VRI with the angle QIR is equal to two right angles (by Prop. 29), but VRX is equal to QIR (by Constr.), therefore VRI with VRX is equal to two right angles (by Ax. 2), therefore IR and RX are in a direct line (by Prop. 14); it can be similarly demonstrated that RX and XL are in a direct line, IL is therefore one direct line, and because QV is parallel to IR, the angle QVR with IRV is equal to two right angles (by Prop. 29), but IR is parallel to VE, and therefore IRV is equal to the angle EVR (by Prop. 29), therefore QVR with EVR is equal to two right angles (by Ax. 2), and therefore QV and EV are in a direct line: it can be similarly demonstrated that VE and EY are in a direct line: therefore QY is one right line, but it is parallel to IL (by Constr.): and because LY and RV are parallel to the same right line XE, LY will be parallel to RV (by Prop. 30), but IQ and RV are parallel, and therefore LY is parallel to IQ (by Prop. 30); therefore LIQY is a parallelogram (by Def. 30), and has an angle LIQ equal to the given one H, and is equal to the given rectilineal figure (by Constr. and Ax. 2).

Cor. 1.—Hence a way appears by which a parallelogram equal to a given rectilineal figure, can be applied to a given line, and have an angle equal to a given one, namely, by applying a parallelogram to the given line, equal to the first triangle.

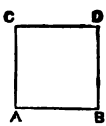
Cor. 2.—A parallelogram equal to two or more rectilineal figures, can be similarly applied to a given line.

Cor. 3.—A parallelogram equal to the difference between two given rectilineal figures, can be found, by applying parallelograms equal to both, to the same right line, and in the same angle, for the difference of these will be a parallelogram, equal to the given difference.

PROPOSITION XLVI. THEOREM.

To describe a square on a given right line (AB).

From an extremity of the given line A, draw AC perpendicular (by Schol. Prop. 11), and equal to AB (by Prop. 3), through C draw CD parallel to AB (by Prop. 31), and through B draw BD parallel to AC. ACDB is a square, described on the given line AB.

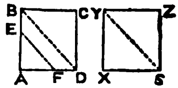


For since ACDB is a parallelogram (by Constr.), and the angle A right (by Constr.), the remaining angles C, D, and B, will also be right angles (by Cor. 1, Prop. 34), and because AC is equal to AB (by Constr.), and the side CD and DB equal to AB and AC (by Prop. 34), the four sides AB, AC, CD and DB will be equal to each other, but all the angles are right angles, and therefore ACDB is a square (by Def. 31).

Cor. 1.—The squares of equal right lines AD and XS are equal.

For draw the diagonals BD and YS, and because the right lines BA, AD are equal to the right lines YX and XS (by Hypoth. and Def. 31), and the angles A and X also equal, the triangles BAD, YXS are equal (by Prop. 4): therefore the squares AC and XZ, which are double of them (by Prop. 34) are equal.

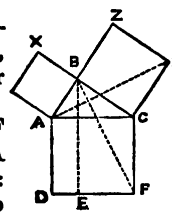
COR. 2.—If two squares AC and XZ be equal, the sides of them will be equal.



For if not, but one (AD) be supposed greater than the other, assume the line AF equal to XS (by Prop. 3), and draw EF. The triangle EAF will be equal to the triangle YXS (by Prop. 4), but YXS is equal BAD, because they are the halves of the equal squares AC and XZ; therefore EAF is equal to BAD (by Ax. 1), a part to the whole, which is absurd; therefore neither AD nor XS is greater than the other, they are therefore equal.

PROPOSITION XLVII. THEOREM.

In any right angled triangle (ABC) the square of the side (AC) opposite to the right angle, is equal to the sum of the squares of the remaining sides (AB and CB).



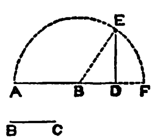
On the sides AB, AC and BC describe the squares AX, AF and BI, and draw BE parallel to either CF or AD, let BF and AI be joined.

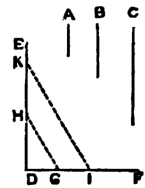
Because the angles ICB and ACF are equal (by Ax. 11), by adding BCA to both, ICA and BCF will be equal: and the sides IC, CA, are equal to the sides BC, CF (by Def. 31); therefore the triangles ICA and BCF are equal (by Prop. 4), but the parallelogram CZ is double of the triangle ICA, because they are between the same parallels CI and AZ, and on the same base CI (by Prop. 41): and the parallelogram CE is double of the triangle BCF, because they are between the same parallels CF and BE, and on the same base CF (by Prop. 41); therefore the parallelograms CZ and CE are double of the equal triangles ICA and BCF, and are therefore equal to each other (by Ax. 6); it can be similarly demonstrated that AX and AE are equal: therefore the whole DACF is equal to the two squares taken together CZ and AX.

Schol.—Let AZ be assumed to be parallel to CI, and thence that the two AB and BZ be in a direct line. This is manifest (by Prop. 14), for the angles CBA and CBZ, on each side of the right line CB, are right.

Cor. 1.—Two sides of a right angled triangle being given, the remaining side can be found, for it is the square root of the sum or difference of the squares of the given lines, according as the given sides be about the right angle, or not; if about it, take the sum, if not, take the difference.

Cor. 2.—Given any number of squares, to find a square equal to all taken together.





Let the right lines A, B and C be sides of the given squares, and construct a right angle FDE, and cut off right lines on the legs of it, equal to the given lines A and B, (by Prop. 3), and draw GH, and assume DI and DK equal to GH, and to the given line C; the square of IK is equal to the squares of A, B and C: for it is equal to the squares of DI and DK (by Prop. 47), but the square of DI is equal to the squares of DG and DH (by Prop. 47), therefore the square of IK is equal to the squares of DK, DH and DG, or of C, B and A, which are equal to DK, DH and DG.

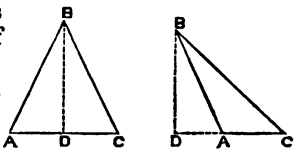
Cor. 3.—Given two unequal right lines AB and BC, to find a right line, whose square is equal to the excess of the square of the greater given line, above the square of the less.

Produce either of the given lines AB, that the produced BD may be equal to the other BC (by Prop. 3), from the centre B, with the greater given line as an interval, describe a semicircle AEF; through D draw DE perpendicular to AD, it will be the right line sought for. For draw BE. The square of BE, or of BA, it

being equal to BE, is equal to the squares of BD and DE (by Prop. 47); if therefore the square of BD be taken away from the square of BA, the remainder will be equal to the square of DE.

Cor. 4.—If from any angle of a triangle ABC, a perpendicular be let fall on the opposite side, the difference of the squares of the sides AB and BC about that angle, will be equal to the difference of the squares of the segments, AD and DC, of the side on which the perpendicular falls.

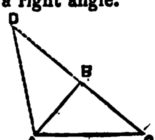
For the square of the side AB is equal to the squares of AD and DB (by Prop. 47), and the square of BC is equal to the squares of BD and DC (by Prop. 47); therefore the difference of the squares of AB and BC, is equal to the difference between the sum of the squares of AD and DB, and the sum of the squares of CD and DB (by Ax. 3); or taking away the common square of DB, is equal to the difference between the squares of AD and DC.



Cor. 5.—The excesses of the squares of each of the sides AB and BC, above the squares of each of the conterminous segments, AD and DC, of the remaining side, are equal. For it appears that the excess of each is equal to the square of the perpendicular (by Prop. 47).

PROPOSITION XLVIII. THEOREM.

If the square of one side (AC) of a triangle (ABC) be equal to the squares described upon the remaining sides (AB and BC), the angle (ABC) opposite to that side, will be a right angle.



To AB one of the sides about the angle ABC, and at the extremity of it B, draw the perpendicular BD (by Schol. Prop. 11), and equal to the other side BC (by Prop. 3), and join AD. The square of AD is equal to the

squares of AB and BD (by Prop. 47), or to the squares of AB and BC, being equal to BD (by Constr.); but the squares of AB and BC are equal to the square of AC (by Hypoth.); therefore the square of AD is equal to the square of AC, and therefore the right lines themselves AD and AC are equal (by Cor. 2, Prop. 46); but DB and BC are also equal, and the side AB is common to both triangles, and therefore the angle ABC is equal to the angle ABD (by Prop. 8); but ABD is a right angle (by Constr.), therefore ABC is also a right angle.

QUESTIONS REFERRING TO THE FIRST BOOK.

What is a finite right line?

How do you bisect a given finite right line?

What are equal in an isosceles triangle?

If two sides of a triangle, and the angle contained by them, be equal to two sides and the contained angle of another, what will be also equal?

By what proposition can you divide an angle into 4 equal parts, 8, 16, &c.?

If one side of a triangle be produced, to what will the external angle be equal?

If you subtract the sum of two angles of a triangle from two right angles, to what will the remainder be equal?

ADDITIONAL DEFINITIONS,

OR EXPLANATION OF TERMS.

1. Geometry is a science which has for its object, the measurement of extent.

2. Extent has three dimensions, length, breadth, and height.

3. The general name of proposition is indifferently ascribed to theorems, problems, and lemmas.

4. An axiom is a proposition, evident by itself.

5. A theorem is a truth which becomes evident by means of a reasoning called demonstration.

6. A problem is a question proposed, which requires a solution.

7. A lemma is a truth employed subsidiarily for the demonstration of a theorem, or the solution of a problem.

8. A corollary is a consequence deduced from one or more propositions.

9. A scholium is a remark on one or more preceding propositions, tending to make known their connection, utility, restriction, or extension.

10. Hypothesis is a supposition, made either in the enunciation, or in the course of a demonstration.

11. The enunciation of a proposition, is that part of it which gives a distinct notion of what we wish to signify or perform, *ex. gr.—To cut a given right line into two equal parts*, is the enunciation of the 10th of the 1st Book.

12. We say that lines are conterminous, when terminating in the same point; and homologous when having the same proportion as that of lines similarly situated.

DEFINITIONS OF ALGEBRA.

1. The mark $=$ is the sign of equality; thus the expression $a=b$ signifies that a equals b.

2. To show that a is less than b, we write $a < b$; and to show that a is greater than b, we write $a > b$.

3. The positive sign $+$ is called plus; it indicates addition.

4. The negative sign $-$ is called minus; it indicates subtraction: thus $a+b$ represents the sum of the quantities a and b; and $a-b$ represents their difference or that which remains in taking b from a; in the same way $a-b+c$, or $a+c-b$, signifies that a and c ought to be joined together, and that b ought to be taken from the whole.

5. The sign $\times$ indicates multiplication; thus $a \times b$ represents the product of a multiplied by b. We also indicate multiplication by a point, as $a.\,b$, or without a point, as $a\,b$, both of which indicate the same thing with $a \times b$. The expression $a \times (b+c-d)$ represents the product of a by the quantity $b+c-d$. If it be necessary to multiply $a+b$ by $a-b+c$, we indicate the product thus $(a+b) \times (a-b+c)$; all that which is contained within the parenthesis is considered as a single quantity. A vinculum is frequently used instead of the parenthesis, thus $\overline{a+b} \times \overline{a-b+c}$. is the same as above.

6. The square of a is indicated by a^2, being the same as $a \times a$, or the second power of a; other powers are shown in the same manner, by placing a figure according to the power, as at a.

7. The square root is indicated as follows $\sqrt[2]{a}$ or $\sqrt{a}$ or $a^{\frac{1}{2}}$, any one of which shows the square root of a; other roots are similarly represented, all the difference being that of the figure, which is called the Index; the cube or third root is thus indicated $\sqrt[3]{a}$ or $a^{\frac{1}{3}}$.

8. This mark $\div$ signifies division, but the usual mode of expression in Algebra is by placing the dividend above

the divisor, thus $\frac{a}{b}$, when a is to be divided by b; and thus $\frac{5}{4}$, when 5 is to be divided by 4.

9. Like quantities are such as b, $4b$, $8b$, and can be incorporated, because one b + four times b + eight times b makes thirteen times b. Likewise $10\,cx$ + $8\,cx$ can be incorporated, which become $18\,cx$, and therefore are like quantities.

10. Unlike quantities are such as $a + b + c$, for a, b, c, would not make three times a, and therefore cannot be incorporated.

11. When we wish to show that $9 \times 10 \times 12$, being numerical, are to be multiplied, we must either retain the $\times$ or dot .; but in order to represent $a \times c \times d$, we can write them thus, $a\,c\,d$. When we write $10\,c\,b\,x\,y$, the number 10 is called the co-efficient of $c\,b\,x\,y$, or $10c$ the co-efficient of $b\,x\,y$, &c.

12. Let it now be understood that $a + b + c$ can be written in no other form, being unlike quantities; but $a + a + a$ is $3a$, which we read three times a. We must also understand that $a + a - a$ is equal to twice a minus one a. Similarly $a \times a \times b \times c$ is similar to $a^2\,b\,c$, by which is meant a multiplied by a multiplied by b multiplied by c; likewise a $^2\sqrt{a}\,b^2$ indicates $a^2 \times \sqrt{a} \times b^2$. When we write $(a + b)^2$ it indicates the square of $a + b$, as all within a parenthesis or beneath a vinculum is considered as a single quantity. One is to be considered the co-efficient of a when no figure is expressed, thus a means $1a$, and when no sign is expressed plus is understood, thus a means $+ a$, but when it should be minus it is always expressed. Again, the expression a^{-2} means $\frac{1}{a^2}$. Similarly, $a^{-\frac{3}{2}}$ means $\frac{1}{\sqrt[2]{a^3}}$, because a^0 always indicates 1, since $1a^0$ becomes 1.

ADDITION OF ALGEBRAIC QUANTITIES.

This rule means the incorporation of similar quantities having like signs, rather than addition. *Ex. gr.* $+ a + a + a - a$ mean that the quantities with like signs, $+ a + a + a$, are to be added, which make $3a$, and that the similar quantity $- a$, with the unlike sign, is to be subtracted from their sum, prefixing to the remainder the sign of the greater quantity; therefore $+ a + a + a - a$ becomes $+ 2a$, or $2a$, as the sign $+$ is understood when no other quantity stands before it.

Rule.—Add all the co-efficients of similar quantities having like signs, and if there be unlike signs, add each and subtract the less sum from the greater, prefixing the sign of the latter before the common letter or letters.

Add together the following Algebraic quantities:

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.
$6ax$	$-4z$	$7x$	$-2xz$
$2ax$	$-5z$	$-3x$	$+3xz$
$7ax$	$-2z$	$+8x$	$-9xz$
$15ax$	$-11z$	$12x$	$-8xz$

Ex. 5.	Ex. 6.
$2a + 3b - 4c$	$3xd + 4bz$
$5a - 2b - 7c$	$2yc + 3md$
$7a + b - 11c$	$3xd + 2yc + 4bz + 3md$

Ex. 7.

$$\begin{array}{l} ay^3 + bx^2 + dx + d \\ 4\,ay^3 - cx^2 - cx - c \\ 2\,ay^3 - dx^2 - ex + h \\ \hline \end{array}$$

$7\,ay^3 + bx^2 - cx^2 - dx^2 + dx - cx - ex + d - c + h$, or,

$7\,ay^3 + (b - c - d)\,x^2 + (d - c - e)\,x + d - c + h$, or,

$7\,ay^3 + \overline{b - c - d}.\,x^2 + \overline{d - c - e}.x + d - e + h$

Ex. 8.

$$\begin{array}{l} 2\,a^2 - 4\,\sqrt{x} + d \\ -\sqrt{x} + a^2 - d \\ \hline 3\,a^2 - 5\,\sqrt{x} \text{ or } 3\,a^2 - 5x^{\frac{1}{2}} \\ \hline \end{array}$$

Ex. 9.

$$\begin{array}{l} 3\,x\,.\,\overline{a + b} - 2 - 2x^{\frac{1}{2}} \\ 5\,\sqrt{x} + 2\,x\,(a + b) - 9 \\ \hline 5\,x\,.\,\overline{a + b} + 3\,3\,\sqrt{x} - 11 \\ \hline \end{array}$$

Ex. 10.

$$\begin{array}{l} 3\,x - 2\,x^{\frac{1}{3}} + 3\,\sqrt{x} \\ 4\,\sqrt{x} - x^3 + x^{\frac{1}{3}} \\ 4\,\sqrt[3]{x} - x + 3\,x \\ \hline 5\,\sqrt[3]{x} + 5\,x^{\frac{1}{2}} + 5\,x - x^3 \\ \hline \end{array}$$

Ex. 11.

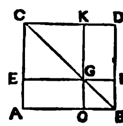
$$\begin{array}{l} \sqrt{x} + x^{\frac{1}{2}} + \sqrt[3]{x} + x^{\frac{1}{3}} \\ -\,x^{\frac{1}{3}} - \sqrt{x} - x^{\frac{1}{2}}\,\sqrt{x} \\ \hline * \quad * \quad * \quad * \\ \hline \end{array}$$

END OF FIRST BOOK.

SECOND BOOK.

DEFINITIONS.

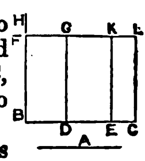
1. Every right angled parallelogram, is said to be contained by the two right lines which make the right angle.



2. In any parallelogram either of these parallelograms, (EK or OF,) which are about the diagonal, together with the two complements, (AG and GD,) is called the gnomon.

PROPOSITION I. THEOREM.

If there be two right lines, (A and BC,) the latter of which (BC) is divided into any number of parts, (BD, DE, EC,) the rectangle under these lines (A and BC,) shall be equal to the rectangles under the undivided line (A,) and the parts of the divided line, (BD, DE, and EC).



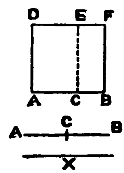
From B erect BH, perpendicular to BC, and in it take BF equal to A, and through F draw FL, parallel to BC, and draw DG, EK, and CL, parallel to BF.

It is evident the rectangle BL is equal to the rectangles BG, DK, and EL, but the rectangle BL, is the rectangle under A and BC; for BF is equal to A; but the rectangles BG, DK, and EL, are the rectangles under A and BD, A and DE, and under A and EC, for each of the lines BF, DG and EK, is equal to A. (by Prop. 34, B. 1.)

COR.—Hence, and from Prop. 34, B. 1, it is evident the area of a rectangle is found, by multiplying the altitude into the base; and from Prop. 35 and 36, B. 1, the area of any parallelogram is found by multiplying the altitude into the base: and from Prop. 37 and 38, B. 1, the area of a triangle is found by multiplying the altitude into half the base.

PROPOSITION II. THEOREM.

If a right line (AB) be divided in any way whatsover (in C,) the square of the whole line (AB) shall be equal to the rectangles under the whole line (AB), and each of the parts (AC, CB).



For, on AB describe the square ADEF (by Prop. 46, B. 1), and from C, draw CE parallel to AD.

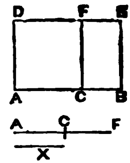
The square AF is equal to the rectangles AE and CF. But the rectangle AE is the rectangle under AB and AC, because AD is equal to AB, (by Const.): and the rectangle CF, is the rectangle under AB and CB, because CE is equal to AB, (by Prop. 34, B. 1).

OTHERWISE.

Assume a right line X equal to AB. The rectangle under X and AB, or the square of AB. (by Hypoth.) is equal to the sum of the rectangles under X and AC, and under X and CB, (by Prop. 1, B. 2,) that is to the sum of the rectangles under AB and AC, and under AB and CB.

PROPOSITION III. THEOREM.

If a right line (AB) be divided in any way (in C), the rectangle under the whole line (AB), and either part (AC), will be equal to the rectangle under the parts (AC and CB), together with the square of the part (AC).



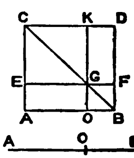
For on AC, describe the square ADFC, and from B, draw BE parallel to AD, until it meet DF produced to E. The rectangle AE is equal to the square ADFC, together with the rectangle CE. But the rectangle AE is the rectangle under AC and AB, for AD is equal to AC, (by Const. and Def. 31, B. 1,) and the square ADFC is the square of AC, (by Const.,) and the rectangle CE is the rectangle under AC and CB, for CF is equal to AC, (by Const., and Defin. 31, B. 1.)

OTHERWISE.

Assume a right line X, equal to AC. The rectangles under X and AF, is equal to the sum of the rectangles under X and AC, and under X and CF, (by Prop. 1, B. 2.) But the rectangle under X and AF, is the rectangle under AC and AF; and the rectangle under X and AC, is the square of AC: and the rectangle under X and CF, is the rectangle under AC and CF.

PROPOSITION IV. THEOREM.

If a right line (AB) be divided into any two parts (in O) the square of the whole line is equal to the squares of the parts, together with twice the rectangle under the parts.



On AB describe the square ACDB, and draw CB, and from O draw OK parallel to AC, and cutting CB in G, and through G draw EF parallel to AB. The square ACDB, is equal to the squares EK and OF, together with the rectangles AG and GD. But OF is the square of OB (by Const., and Cor. of Prop. 43, B. 1,) and because EG is equal to AO (by Const. and Prop. 34, B. 1,) EK is the square of AO, and because OG and OB are equal, (by Defin. 31, B. 1,) AG is the rectangle under the parts AO and OB, but GD is equal to AG, (by

Prop. 43, B. 1,) therefore AG and GD together are equal to twice the rectangle under the parts.

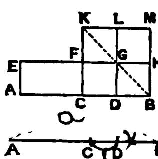
OTHERWISE.

The square of AB is equal to the sum of the rectangles under AB and AO, and under AB and BO, (by Prop. 2, B. 2,) but the rectangle under AB and AO, is equal to the sum of the rectangle under AO and OB, and the square of AO, (by Prop. 3, B. 2); and the rectangle under AB and BO, is equal to the sum of the rectangle under AO and OB, and the square of OB (by last ref.); therefore, the sum of the rectangles under AB and OA, and under AB and BO, (or the square of AB,) is equal to the sum of the squares of AO and OB, and twice the rectangle under AO and OB.

COR.—Hence it appears that the square of half the line is the fourth part of the square of the whole; for, a right line being bisected, the rectangle under the parts is equal square of the half.

PROPOSITION V. THEOREM.

If a right line (AB) be divided into equal parts (in C), and into unequal parts (in D), the rectangle under the unequal parts (AD and DB,) together with the square of the intermediate part (CD) will be equal to the square of half the line (CB).



Describe on CB the square CKMB, and draw KB, and from D draw DL, parallel to CK, and cutting KB in G, and through G draw HGE parallel to AB, meeting AE drawn from A parallel to CK.

Because AC and CB are equal (by Hypoth.), the rectangles AF and CH are equal (by Prop. 36. B. 1), but the rectangles CG and GM are equal (by Prop. 43, B 1), therefore the rectangle AG, is equal to the Gnomon CHL (by Ax. 2); to both let

FL be added, and the rectangle AG with the square FL, will be equal to the square CKMB. But AG is the rectangle under AD and DB, for DG is equal to DB (by Cor. Prop. 43. B. 1, and Def. 31. B. 1); and FL is the square of CD, because FG is equal to CD (by Prop. 34. B. 1), and CKMB is the square of CB.

OTHERWISE.

The rectangle under AD and DB is equal to the sum of the rectangles under AC and DB, and under CD and DB. (by Prop. 1. B. 2); but the rectangle under AC and DB is equal to the rectangle under CB and DB (because AC and CB are equal), or to the rectangle under CD and DB, with the square of DB (by Prop. 3. B. 2), therefore the rectangle under AD and DB is equal to twice the rectangle under CD and DB together with the square of DB; add to both the square of CD, and the rectangle under AD and DB together with the square of CD, is equal to twice the rectangle under CD and DB, together with the squares of CD and DB, which is the square of CB (by Prop. . B. 2).

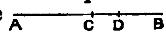
Cor. 1.—Hence it appears if a line be bisected, the rectangle under the parts is greater than if it be divided unequally, and therefore the sum of the squares of the parts is less (by Prop. 4. B. 2.)

Cor. 2.—If two equal right lines be so divided, that the rectangle under the segments of one, be equal to the rectangle under the segments of the other, the segments themselves shall be equal.

If right lines be bisected, the thing is evident. But if they be not bisected, let there be AB and CD, and divide them in E and F. Let them be bisected in G and H, and because the right lines themselves are equal, (by Hypoth.), the halves of them will be equal, and therefore the squares of their halves (by Cor. 1. Prop. 46. B. 1), but the rectangles under AE and EB, and under

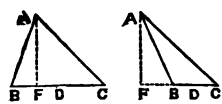
CF and FD are also equal (by Hypoth.), therefore taking away these from the equal squares, the residue shall be equal, namely, the squares of GE and of HF (by Prop. 5. B. 2), and the lines themselves GE and HF are equal (by Cor. 2. Prop. 46. B. 1), therefore the sums and differences of them and the halves are equal, namely, the segments AE and CF, EB and FD.

Cor. 3.—The rectangle under the sum and difference of two lines, is equal to the difference of the squares of these lines. Because the said rectangle together with the square of the less, is equal to the square of the greater line, as appears from the preceding proposition, for AC is the greater line, CD the less, and DB the difference.



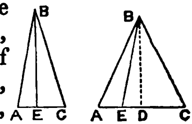
Schol.—Hence it appears that in a right angled triangle, the rectangle under the sum and difference of the hypothenuse and one side, is equal to the square of the remaining side.

Cor. 4.—The difference of the squares of two sides BA and AC, of any triangle BAC, is equal to twice the rectangle under the remaining side BC, and the distance of the perpendicular (let fall from the opposite angle A) from the middle point D. Because the difference of the squares of the sides BA and AC, is equal to the difference of the squares of the segments, BF and FC, of the side on which the perpendicular falls (by Cor. 4. Prop. 47. B. 1), and therefore when the perpendicular falls inside the triangle, to the rectangle under BC, the sum of these segments, and the difference between BF and FC (by Cor. 3. Prop. 5. B. 2), or to twice the rectangle under BC and half the difference, namely the distance of the point F from the middle D of the side BC. But if the perpendicular falls outside the triangle, the difference of the squares of the segments BF and FC, will be equal to the rectangle under



their difference BC, and the sum of them BF and FC together, or to twice the rectangle under BC and half of BF and FC, namely the distance of the point F from the middle point D of the side BC.

Cor. 5.—If from the vertex of an Isosceles triangle ABC, a right line BE, be drawn to AC, the square of BE, with the rectangle under the segments of the base AE and EC, will be equal to the square of one of the other sides AB or BC. First, let BE be perpendicular to the base, and then AC will be bisected (by Cor. 1. Prop. 26. B. 1), therefore the rectangle under AE and EC, is equal to the square of AE; but the square of AE, with the square of EB, is equal to the square of AB (by Prop. 47. B. 1), and therefore the rectangle under AE and EC, with the square of EA, is equal to the square of AB.



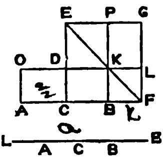
Then, let BE not be perpendicular to the base, and draw BD perpendicular, therefore AC will be bisected in D (by Cor. 1. Prop. 26. B. 1), and divided unequally in E, and in like manner the rectangle under CE and EA, with the square of DE, is equal to the square of DA (by Prop. 5. B. 2), add to both the square of BD, and the rectangle under CE and EA, with the squares of DE and BD, or with the square of BE, (by Prop. 47. B. 1), will be equal to the squares of BD and DA, or to the square of BA.

PROPOSITION VI. THEOREM.

If any right line (AB) be bisected (in C) and another right line (BF) be added in directum, the rectangle under the whole (AF), (composed of AB and BF), and the added part (BF) together with the square of the half (CB) will be equal to the square of (CF), the line composed of the half and added part.

For on CF describe a square CEGF, and draw EF, and from B draw BP parallel to FG, dividing EF in K,

and through K, draw LO, parallel to CF, meeting AO drawn from A, parallel to CD; because AC and CB are equal (by Hypoth.), AD is equal to CK (by Prop. 36. B. 1); but CK and KG are equal (by Prop. 43. B. 1), therefore AD is equal to KG; add CL to both, and the gnomon CLP is equal to AL, to both add DP, and AL and DP together, will be equal to the square of CF. But AL is the rectangle under the whole line produced and the produced part, for FL is equal to BF (by Cor. Prop. 43. B. 1, and Def. 31. B. 1), and DP is the square of the half CB, for it is the square of DK (by Cor. Prop. 43. B. 1), and DK is equal to CB (by Prop. 34. B. 1.)



OTHERWISE,

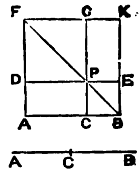
Let there be added to each extremity of the given line AB, the equal right lines LA and BE, and LE will be bisected in C, and divided unequally in B, and therefore the rectangle under LB and BE, together with the square of CB will be equal to the square of CE (by Prop. 5. B. 2), but the rectangle under LB and BE is equal to the rectangle under AE and EB, because LA and BE are equal.

PROPOSITION VII. THEOREM.

If a right line (AB) be cut in any two parts, (in C) the squares of the whole line (AB) and one segment (CB) taken together, will be equal to twice the rectangle under the whole line (AB) and said segment (CB) together with the square of the other segment (AC).

On AB describe a square, and draw FB, from C draw CG parallel to AF, and through P (the intersection of it with FB), draw DE parallel to AB.

The square AK is equal to the rectangles AE and PK together with the square DG, add to both the square



CE, and the square AK and CE together, are equal to the rectangles AE and CK together with the square DG. But AE is equal to the rectangle under AB and CB, because CB and BE are equal (by Cor. Prop. 43. B. 1), and CK is also equal to the rectangle under AB and CB, because KB is equal to AB (by Def. 31. B. 1), and DG is the square of AC, because DP and AC are equal, (by Prop. 34. B. 1.)

OTHERWISE,

The square of AB is equal to twice the rectangle under AC and CB, together with the squares of AC and CB, add to both the square of CB, and the square of AB together with the square of CB, is equal to twice the rectangle under AC and CB together with the square of CA and twice the square of CB; but twice the rectangle under AC and CB together with twice the square of CB, is equal to twice the rectangle under AB and BC (by Prop. 3. B. 2); therefore the square of AB together with the square of CB, is equal to twice the rectangle under AB and BC together with the square of AC.

Cor.—Hence it appears that the excess of the sum of the squares of the two right lines AB and CB, above the double rectangle under them AB and BC, is equal to the square of the difference between them, AC.

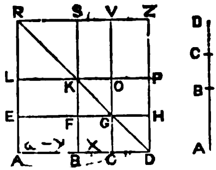
PROPOSITION VIII. THEOREM.

If a right line (AC) be cut into any two parts (in B), the rectangle under the whole line (AC) and either of the other parts (BC) four times taken, together with the square of the remaining part (AB), will be equal to the square, as if on one right line, which is described on the whole line (AC) and said part (BC).

Produce AC till CD is equal to BC (by Pos. 1. and Prop. 3. B. 1), and on the right line AD describe the

$4ax + (a-x)^2 = (a+x)^2$

square ARZD (by Prop. 46. B. 1), and from B and C draw BS and CV parallel to AR; and draw RD, through G and K draw EH and LP parallel to AD. Because BC and CD are equal (by Constr.) and SV is equal to BC (by Prop. 34. B. 1), and VZ to CD, SV and



VZ are also equal, and therefore SG and VH are equal (by Prop. 36. B. 1); but VH and AG are equal, and therefore SG is equal to AG; and because FG is equal to BC (by Prop. 34. B. 1), FG and CD are equal, and therefore the square FO is equal to the square CH: and also EK and KV are equal (by Prop. 43. B. 1), therefore if to these equals be added the equals CH and FO, EK and CH together will be equal to SG, and therefore to AG; therefore AG, SG, and VH together with EK and CH are four times AG; but AG, SG, and VH together with EK and CH and the square LS, are equal to the square AZ, therefore AG four times taken together with LS, is equal to AZ.

But AG is the rectangle under AC and BC, because CG is equal to CD (by Cor. Prop. 43. B. 1), and therefore to BC (by Constr.); and LS is the square of AB, because AB and RS are equal (by Prop. 34. B. 1.)

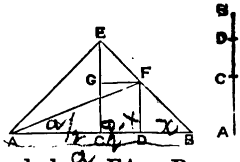
OTHERWISE,

Produce AC until CD is equal to BC; the square of AD is equal to the squares of AC and CD, together with twice the rectangle under AC and CD, that is, because BC and CD are equal (by Constr.) to the squares AC and BC, together with twice the rectangle under AC and CB; but the squares of AC and BC are equal to twice the rectangle under AC and CB together with the square of AB (by Prop. 7. B. 2), therefore the square of AD is equal to the rectangle under AC and CB four times taken together with the square of AB.

$(a-x)^2+x^2 = 2\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}-x\right)^2$

PROPOSITION IX. THEOREM.

If a right line (AB) be divided into equal parts (in C) and in unequal parts (in D) the squares of the unequal parts (AD and DB) will be double the square of the half (AC) and of the square of the intermediate part (CD) taken together.



From the point C draw CE perpendicular to AB, and equal to either AC or CB (by Prop. 11. and 3. B. 1), and draw AE and EB, from D draw DF parallel to CE, and from F draw FG parallel to CD, and draw FA. Because the angle ACE is a right angle, and the sides AC and CE are equal (by Constr.), CEA is half a right angle (by Cor. 3. Prop. 32. B. 1.) It may be similarly demonstrated that CEB is half a right angle, therefore AEB is right; and because of the parallels GF and CD, EGF is equal to ECB (by Prop. 29. B. 1), therefore EGF is also a right angle, but GEF is half a right angle, therefore GFE is also half a right angle, and GE and GF are equal; (by Prop. 6. B. 1); in the same manner the angle FDB is right, because it is equal to the angle ECB on account of the parallels FD and CE, but DBF is half a right angle, and therefore DFB is half a right angle, and the sides DF and DB are equal (by Prop. 6. B. 1); therefore because AC and CE are equal, and the angle ACE right, the square of AE is double of the square of AC; and because EG and GF are equal, and the angle EGF right, the square of EF is double of the square GF, but GF and CD are equal (by Prop. 34. B. 1), therefore the square of EF is double of the square of CD; therefore the squares of AE and EF are double of the squares of AC and CD; but because the angle AEF is right, the square of AF is equal to the squares of AE and EF (by Prop. 47. B. 1), therefore the square of AF is double of the squares of AC and CD, but the square

of AF is equal to the squares of AD and DF, because the angle ADF is right (by Prop. 47. B. 1), therefore the squares of AD and DF, are double of the squares of AC and CD, but DF and DB are equal, and therefore the squares of AD and DB are double the squares of AC and CD.

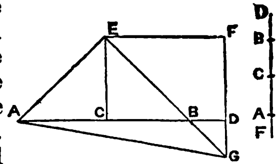
OTHERWISE,

The square of AD is equal to the squares of AC and CD, with twice the rectangle under AC and CD (by Prop. 4. B. 2), or because AC and CB are equal, with double the rectangle under BC and CD; add to both the square of DB, and the squares of AD and DB are equal to the squares of AC and CD and DB, with double the rectangle under BC and CD; but twice the rectangle under BC and CD with the square of DB, is equal to the squares of CB and CD (by Prop. 7. B. 2), or because AC and CB are equal to the squares of AC and CD; and therefore the squares of AD and DB, are equal to twice the square of AC with twice the square of CD.

PROPOSITION X. THEOREM.

If any right line (AB) be bisected (in C) and any right line (BD) be added, in directum, the square of the whole line (AD) together with the square of the added part (BD) will be equal to twice the square of the line (CD) which is composed of the half and added part, together with twice the square of the half (AC).

From the point C draw CE perpendicular to AB, and equal to either CA or CB (by Prop. 11. B. 1), and draw AE, from E draw EF parallel to AB (by Prop. 31. B. 1), and from D draw DF parallel to CE; and because the angles CEF and DFE are equal to two right angles (by Prop. 29. B. 1) on account of the parallels CE and DF, the angles BEF and DFE are less than two right angles, therefore the right lines EB and FD shall meet (by Ax. 12.), let them then meet in G, and draw GA.



Because CA and CE are equal (by Constr.), and the angle C a right angle (by Constr.), the angle CEA is half a right angle (by Cor. 3. Prop. 32. B. 1); similarly it is proved that CEB is half a right angle, therefore the angle AEB is right; and because DG and EC are parallel (by Constr.), the alternate angles GDB, ECB are equal, and therefore GDB is right; but the angles DBG, EBC are equal (by Prop. 15. B. 1), and EBC is half a right angle; therefore DBG is half a right angle, and for the same reason DGB is half a right angle, therefore the sides DB, DG are equal (by Prop. 6. B. 1), and because EGF is half a right angle, and the angle at F right, for it is opposite the equal angle C (by Prop. 34. B. 1), FEG will be half a right angle, and therefore the sides EF and FG are equal.

Therefore because AC and CE are equal, and the angle ACE right, the square of AE is twice the square of AC, and because GF and FE are equal, and the angle at F right, the square of GE is equal to twice the square EF; but EF and CD are equal (by Prop. 34 B. 1), therefore the square of GE is equal to twice the square of CD; but the square of AE is also twice the square of AC; therefore the squares of AE and EG are double the squares of AC and CD; but the square of AG is equal to the squares of AE and EG, (by Prop. 47. B. 1,) and therefore equal to twice the squares of AC and CD; and the squares of AD and DG are equal to the square of AG, and therefore are equal to twice the squares of AC and CD; but BD and DG are equal, and therefore the squares of AD and DB, are double the squares of AC and CD.

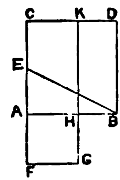
OTHERWISE,

To the given line AB, add FA equal to BD; the right line FD is therefore bisected in C, and cut unequally in

B, and therefore the squares of FB and BD are double of the squares of FC and CD (by Prop. 9. B. 2); but FB and AD are equal, and also FC and CD, because FA is equal to BD; therefore the squares of AD and DB are double of the squares of CD and CB or AC.

PROPOSITION XI. PROBLEM.

To divide a given finite right line (AB) so that the rectangle contained under the whole line and one part, shall be equal to the square of the other part.



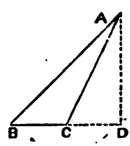
From A draw AC, perpendicular to the given line AB, and let AC be equal to AB and bisect it in E, and draw EB, produce CA until EF is equal to EB, and in the given line AB assume AH equal to AF; the square of which is equal to the rectangle under the remaining segment HB and the whole line AB. And complete the square of AB, and through H draw the right line GK parallel to AC, and from F the right line FG parallel to AB. Because CA is bisected in E, and AF is added, the rectangle under CF and FA, together with the square of EA, is equal to the square of EF (by Prop. 6. B. 2), or to the square of EB, which is equal to EF (by Constr.), and therefore to the squares of EA and AB (by Prop. 47. B. 1); take away the common square of EA, and the rectangle under CF and FA, will be equal to the square of AB, but since AF and FG are equal, CG is the rectangle under CF and AF, therefore CG and AD are equal, and if the common rectangle CH be taken away, AG and HD are equal, but AG is the square of AH, for AF and AH are equal (by Constr.), and the angle A is right; and HD is the rectangle under AB and HB, for BD is equal to AB.

PROPOSITION XII. THEOREM.

In any obtuse angled triangle (as BAC) the square of the side (AB) subtending the obtuse angle, is greater than the

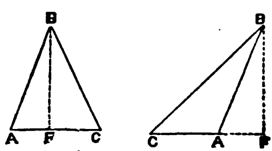
squares of the sides (BC and CA) which contain the obtuse angle, by twice the rectangle contained under either of these sides (BC), and the external part (CD) intercepted between the perpendicular let fall from the opposite angle A, and the obtuse angle.

The square of BA is equal to the squares of AD and DB (by Prop. 47. B. 1), but the square of DB is equal to the squares of DC and CB, together with twice the rectangle under DC and CB (by Prop. 4. B. 2), and therefore the square of AB is equal to the squares of AD, DC, and CB, together with twice the rectangle under DC and CB; but the square of AC is equal to the squares of AD and DC, (by Prop. 47. B. 1), and therefore the square of AB is equal to the squares of AC and CB, together with twice the rectangle under DC and CB; therefore the square of AB is greater than the squares of AC and CB, by twice the rectangle under DC and CB.



PROPOSITION XIII. THEOREM.

In any triangle (ABC) the square of the side (AB) subtending an acute angle, is less than the squares of the sides (AC and CB) containing that angle, by twice the rectangle under either of them (AC) and the part of it intercepted between the perpendicular (BF) let fall from the opposite angle, and the acute angle.

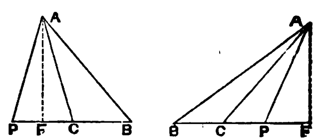


The squares of AC and CF are equal to twice the rectangle under AC and CF, together with the square of AF (by Prop. 7. B. 2), add to both the square of the perpendicular BF, and the squares of AC, CF, and BF, will be equal to twice the rectangle under AC and CF, together with the squares of BF and AF, or with the square of AB which is equal to them (by Prop. 47. B. 1), but the squares of BF and CF are equal to the

square of BC (by Prop. 47. B. 1), and therefore the squares of BC and AC are equal to twice the rectangle under AC and CF, together with the square of AB; therefore the square of AB is less than the squares of AC and CB, by twice the rectangle under AC and CF.

SCHOL. 1.—If the angle CAB be a right angle, the angles A and F coincide and the rectangle under AC and CF will be the square of AC, but it appears in this case that the square of AB is less than the squares of AC and CB by twice the square of AC (by Prop. 47. B. 1.)

SCHOL. 2.—Hence, given the sides of any triangle (in numbers), we are able to find its area, for subtract the square of either of the sides AB, which is not the greatest, from the sum of the squares of the remaining sides AC and CB, and by either of these sides AC divide half the remainder, namely, the rectangle under AC and CF; subtract the square of the quotient CF, from the square of the remaining side CB, and the square root, BF, of the residue, multiplied into half the former divisor AC, gives the area of the triangle.



COR.—If from any angle (A) of a triangle (BAP,) a right line (AC) be drawn, bisecting the opposite side, the squares of the sides containing that angle (AB and AP) will be double of the squares of the bisecting line (AC), and of half the side opposite that angle (A.)

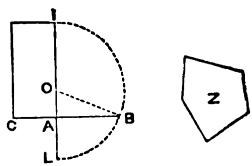
If the bisecting right line be perpendicular to the side, it is evident from Prop. 47. B. 1.

But if not, from the angle A draw to the opposite side the perpendicular AF, and as one of the angles ACB and ACP is obtuse, let ACB be obtuse, and therefore the square of AB is equal to the squares of AC and of CB, together with twice the rectangle under BC and CF, (by Prop. 12. B. 2), but the angle ACP is acute, and therefore the square of AP, together with twice the rect-

angle under PC and CF, or because PC and BC are equal (by Hypoth.), with twice the rectangle under BC and CF (by Prop. 13. B. 2) is equal to the squares of AC and of CP; therefore the squares of AB and AP with twice the rectangle under BC and CF are equal to twice the square of AC, with the squares of BC and CP, and with twice the rectangle under BC and CF, take away from both twice the rectangle under BC and CF; and the squares of AB and AP are equal to twice the square of AC, with the squares of BC and CP, or with twice the square of BC, because BC and CP are equal.

PROPOSITION XIV. PROBLEM.

To construct a square equal to a given rectilineal figure (Z.)



Construct a rectangular parallelogram CI equal to Z (by Prop. 45. B. 1), and if the adjacent sides are equal, the proposition is done. If not, produce either side IA, and make the produced part AL equal to the adjacent side AC (by Prop. 3. B. 1); bisect IL in O, and from the centre O, and with the interval OL describe a semicircle LBI, and produce CA till it meets the periphery in B: the square described on AB, will be equal to the given rectilineal figure. For draw OB; and because IL is bisected in O, and cut unequally in A, the rectangle under IA and AL, together with the square of OA, shall be equal to the square of OL (by Prop. 5. B. 2) or of OB, which is equal to OL, and therefore to the square of OA and AB (by Prop. 47. B. 1); take away from both the square OA, and the rectangle under IA and AL is equal to the square of AB, but the rectangle under IA and AL is equal to IC, for AL and AC are equal (by Constr.); therefore the square of AB is equal to the rectangle IC, and therefore to the rectilineal figure Z.

SUBTRACTION OF ALGEBRA.

This rule being the reverse of Addition, we change the signs of the quantities to be subtracted, or consider them as changed; that is, make negative signs affirmative, and affirmative signs negative; and then incorporate them as in Addition.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.
$5a$	$10d$	$-\ 8x$	$-\ 8z$
$2a$	$9d$	$-\ 2x$	$-\ 7x$
$3a$	d	$-\ 6x$	$-\ 8z + 7x$

In Ex. 1, we have $2a$ to be subtracted from $5a$, and therefore, proceeding according to the Rule, $2a$ becomes $-\ 2a$; then, by subtracting the similar quantities having unlike signs, and prefixing the sign of the greater, as in Addition, we have $3a$ for the result.

Ex. 2 and 3 are combined in the same manner; but Ex. 4, being composed of unlike quantities, requires to be written down as in Addition, with the exception of changing the sign of the quantity to be subtracted.

Ex. 5.	Ex. 6.	Ex. 7.
$10a^2 + 23dc$	$10a^2 + 23dc$	$-\ 10a^2 - 23dc$
$7a^2 + 19dc$	$-\ 7a^2 - 19dc$	$7a^2 + 19dc$
$3a^2 + 4dc$	$17a^2 + 42dc$	$-\ 17a^2 - 42dc$

Ex. 8.	Ex. 9.
$\sqrt{a} - \sqrt{10} + \sqrt{d}$	$20\sqrt{ac} - 2\sqrt{dx}$
$-\sqrt{a} + \sqrt{10} + \sqrt{d}$	$10\sqrt{ac} - \sqrt{dx}$
$2\sqrt{a} - 2\sqrt{10}$ *	$10\sqrt{ac} - \sqrt{dx}$

Ex. 10.

$$\begin{array}{r} 20\sqrt{ac} - 2\sqrt{dc} \\ 8\sqrt{b} - 2\sqrt{x} \\ \hline 20\sqrt{ac} - 8\sqrt{b} - 2\sqrt{dc} + 2\sqrt{x} \end{array}$$

The answers of Ex. 8, 9, 10, may be also written thus :—Ex. 8, $2a^{\frac{1}{2}} - 2.\ 10^{\frac{1}{2}}$ * Ex. 9, $10\ a^{\frac{1}{2}}c - d^{\frac{1}{2}}.x$ Ex. 10, $20a^{\frac{1}{2}}c - 8.b^{\frac{1}{2}} - 2.d^{\frac{1}{2}}c + 2.x^{\frac{1}{2}}$

In Ex. 9, it is evident that once the root of $d.x$ taken from twice the root of $d.x$ leaves once the root of $d.x$. And in Ex. 10, it is evident that the quantities being unlike, must be written as in addition, with the exception of changing the signs of the subtrahend.

Parentheses are used with much advantage, when any number of quantities are multiplied by the same co-efficient. Ex. gr., $ax + cx - dex$, may be written thus, $(a + c - de)x$; and any quantities having common multiplier can be similarly written; which facilitates calculations considerably.

Ex. 11.

$$\begin{array}{r} 3a\sqrt{(x^2 + y)} - 6a^2x + 12 \\ 12 - 5a\sqrt{(x^2 + y)} - 8a^2x \\ \hline 8a\sqrt{(x^2 + y)} + 2a^2x. \quad * \\ \text{or } 8a\sqrt{x^2} + 8a\sqrt{y} + 2a^2x \end{array}$$

Ex. 12.

$$\begin{array}{c} ax^3 + bx^2 + cx^2y - m \\ ex^3 - x^2n + gx^2y - p \\ \hline x^3(a - e) + x^2(b + n) + x^2y(c - g) - m + p \\ \text{or } ax^3 - ex^3 + bx^2 + nx^2 + cx^2y - gx^2y - m + p \end{array}$$

Ex. 13.

$$\overline{a+c}\,.x^2 + ex - d$$
$$-\ bx^2 - ex - d$$

$$(a + c + b)x^2 + 2ex\ *$$
$$\text{or } ax^2 + cx^2 + bx^2 + 2ex\ *$$

Ex. 14.

$$ax^3 - px^3 + qx^2$$
$$ax^3 - px^3 - a^2x^2 + apx^2$$

$$*\quad *\quad (a^2 + q - ap)x^2$$
$$\text{or } *\quad a^2x^2 + qx^2 - apx^2$$

Whenever a negative sign stands before a parenthesis, it affects each sign within it: so that if we have, $-(a + b - c)x$, each of the signs contained within the parenthesis requires to be changed whenever we remove it, thus, $-(a + b - c)x = - ax - bx + cx$; and again, if we wish to place this in a parenthesis, it is also required that we should change the signs, thus. $- ax - bx + cx$ becomes $-(a+ b - c)x$.

Ex. 15.

$$ax^3 + px^2 - bx + r$$
$$mx^3 - nx^2 + dx - c$$

$$(a - m)x^3 + (p + n)x^2 - (b + d)x + c + r$$
$$\text{or } ax^3 - mx^3 + px^2 + nx^2 - bx - dx + c + r$$
$$\text{or } \overline{a - m}\,.x^3 + \overline{p + n}\,.x^2 - \overline{b + d}\,.x + c + r.$$

Ex. 16.

$$ax + bx + c^{\frac{1}{2}}x - d^{\frac{2}{3}}y - m^2y$$
$$px - rx + e^{\frac{1}{2}}x - ey - m^{\frac{2}{3}}y$$

$(a - p + b + r + c^{\frac{1}{2}} - e^{\frac{1}{2}})x - (d^{\frac{2}{3}} - e + m^2 -$
$\quad m^{\frac{2}{3}})y.$

or $ax - px + bx + rx + c^{\frac{1}{2}}x - e^{\frac{1}{2}}x - \sqrt[3]{d^2}y$
$\quad + ey - m^2y + m^{\frac{2}{3}}y,$

or $\overline{a - p + b + \sqrt{c} - \sqrt{e}}.x - \overline{d^{\frac{2}{3}} - e + m^2}$
$\quad \overline{- \sqrt[3]{m^2}}.y.$

END OF SECOND BOOK.

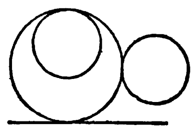
THIRD BOOK.

DEFINITIONS.

1. Equal circles are those of which the diameters are equal.

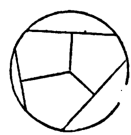
2. A right line is said to touch a circle, which meeting a circle and being produced, does not cut it.

3. Circles are said to touch each other, which meeting do not cut.

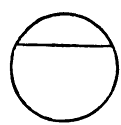


4. In a Circle, right lines are said to be equally distant from the centre, when perpendiculars drawn from them to the centre, are equal.

5. And that line is said to be at the greatest distance from the centre, on which the greatest perpendicular falls.

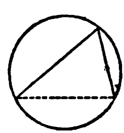


6. A Segment of a Circle, is a figure which is contained by a right line and a part of a circumference.

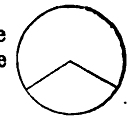


7. An Angle in a Segment, is an angle under right lines drawn from any point, in the part of the circumference, which contains the segment, to the extremities of the segment.

8. An Angle is said to stand upon the part of the circumference, or arch, which is intercepted between its legs.



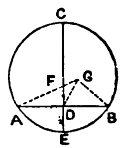
9. A Sector of a circle, is the figure which is contained by two radii and the part of the circumference between them.



10. Similar segments of circles are those which contain equal angles.

PROPOSITION I. PROBLEM.

To find the centre of a given circle (ACB.)

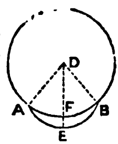


Draw any right line AB, bisect it in D (by Prop. 10. B. 1); from D draw DC perpendicular to AB (by Prop. 11. B. 1), which produce to E; bisect CE in F (by Prop. 10. B. 1), and F will be the centre. But if not, if it be possible to do it, let G be the centre, and draw GA, GD and GB.

Therefore, because in the triangles GDA, GDB, the side GA is equal to the side GB (by Hypoth. and Def. 15. B. 1), and DA to DB (by Constr.), and GD common to both, the angles GDA and GDB are equal (by Prop. 8. B. 1), and are therefore right angles (by Def. 11. B. 1): but the angle CDB is a right angle (by Constr.), therefore GDB is equal to CDB (by Ax. 11), a part to the whole, which is absurd: G therefore is not the centre of the circle ACB, and it can be similarly proved, that no other point, except F, is the centre: therefore F is the centre of the circle ACB.

PROPOSITION II. THEOREM.

If in the circumference of a circle any two points (A and B) be assumed, the right line which joins them falls within the circle.



But, if not, if it be possible, let AEB be a right line, whose point E falls without the circle, and let D be the centre of the circle, and draw DE, DA and DB.

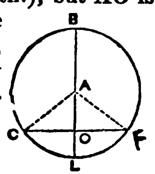
Therefore, because in the triangle ADB, the sides DA and DB are equal (by Def. 15. B. 1), the angle DBA is equal to DAB (by Prop. 5. B. 1); but the external angle DEA is greater than the internal angle DBA (by Prop. 16. B. 1), and therefore is greater than the angle DAB; and therefore the side DA is greater than the side DE (by Prop. 19. B. 1), but the right line DF is equal to DA (by Def. 15. B. 1), and therefore is greater than DE, a part than the whole, which is absurd: therefore the line AEB is not a right line; and it can be similarly demonstrated, that a line, any point of which is on the circumference, is not a right line. Therefore every point of the right line falls within the circle.

PROPOSITION III. THEOREM.

Part 1.—If in a circle, a right line (BL) drawn through the centre divides any line (CF) into two equal parts, which does not pass through the centre, it will be perpendicular to it.

Part 2.—And if it cuts it perpendicularly, it will divide it into two equal parts.

Part 1.—For draw AC and AF; in the triangles AOC, AOF, the side AC will be equal to AF (by Def. 15. B. 1), and also OC to OF (by Hypoth.), but AO is common to both, therefore the angle AOC is equal to AOF (by Prop. 8. B. 1), and therefore each is a right angle (by Def. 11. B. 1), therefore BO is perpendicular to CF (by Def. 11. B. 1.)



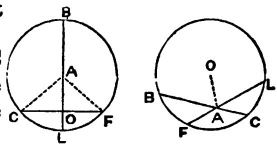
Part 2.—Because in the triangle CAF, the side AC is equal to the side AF (by Def. 15. B. 1), the angle AFC is also equal to the angle ACF (by Prop. 5. B. 1), and therefore in the triangles CAO, FAO, the angle AFO is equal to the angle ACO;

and AOC and AOF are also equal (by Hypoth.), and the side AO opposite the equal angles ACO, AFO, is common to both triangles, and therefore the side OC is equal to the side OF, (by Prop. 26. B. 1), and therefore the right line CF is bisected.

PROPOSITION IV. THEOREM.

If in a circle two right lines cut each other, not both passing through the centre, they do not cut each other into two equal parts.

If one of them BL pass through the centre, and the other CF not, it appears that the first cannot be bisected.

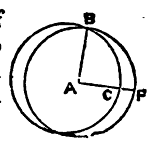


But if neither of them BC and FL pass through the centre, draw OA from the centre to the intersection, and if BC be bisected in A, OA will be perpendicular to it (by Prop. 3. B. 3), and therefore the angle OAC is right; and if FL be bisected in A, OA is perpendicular to FL (by Prop. 3. B. 3), and therefore the angle OAL is right, and therefore equal to OAC (by Ax. 11), a part to the whole, which is absurd: therefore the right lines BC and FL do not cut each other into two equal parts.

PROPOSITION V. THEOREM.

If two circles (ABC, ABF) cut each other, their centres will not be the same.

For, if it be possible, let A be the centre of both circles, and draw AB to the intersection, and also draw ACF where the circles do not intersect.



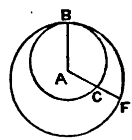
Therefore, because A is the centre of the circle ABC, AB shall be equal to AC (by Def. 15. B. 1); and because A is the centre of the circle ABF, AB will be equal to AF (by Def. 15. B. 1); therefore AC is equal to AF (by Ax. 1),

a part to the whole, which is absurd; therefore A is not the centre of both circles, and it can similarly be proved, that no other point is the centre of both circles.

PROPOSITION VI. THEOREM.

If two circles (ABC, ABF) touch, one being within the other, they will not have the same centre.

For, if it be possible, let A be the centre of both circles, and draw AB to the point of contact, and also draw ACF where they do not cut.



Therefore, because A is the centre of the circle ABC, AB shall be equal to AC (by Def. 15. B. 1); and because A is the centre of the circle ABF, AB will be equal to AF (by Def. 15. B. 1); therefore AC is equal to AF (by Ax. 1), a part to the whole, which is absurd: therefore A is not the centre of both circles, and it may be similarly shown, that no other point is the centre of both circles.

PROPOSITION VII. THEOREM.

1st.—If a point be taken within a circle, which is not the centre, and from it right lines be drawn to the circumference, the greatest will be that which passes through the centre.

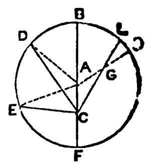
2ndly.—The remaining part of the diameter will be the least.

3rdly.—Those lines which make equal angles with the diameter are equal.

4thly.—That line which is nearer to the diameter is greater than those more remote.

5thly.—More than two lines cannot be drawn equal.

Part 1.—CB passing through the centre, is greater



than any other line CD. For draw AD, and AB is equal to AD (by Def. 15. B. 1), therefore if the common line CA be added to both, CA and AD together will be equal to CB, but CA and AD together are greater than CD (by Prop. 20. B. 1), and therefore CB is greater than CD.

PART 2.—The remaining part of the diameter CF, is less than any other line CE. For draw AE, and AC and CE together are greater than AE (by Prop. 20. B. 1), and therefore greater than AF; take away from both the common part AC, and CE will be greater than CF.

PART 3.—The right lines CL and CD which make equal angles with CB are equal. But if not, if it be possible, let either of them CL be greater, and make CG equal to CD, and draw AD and AG. Therefore in the triangles ACG and ACD, the side AC is common, and CG is equal to CD (by Constr.), and the angle ACG is equal to the angle ACD (by Hypoth.), and therefore the sides AG and AD are equal (by Prop. 4. B. 1), but the right line AD is equal to AO, and therefore AG is equal to AO, a part equal to the whole, which is absurd. Therefore neither CL nor CD is greater than the other, and therefore they are equal.

PART 4.—CD or CL which is nearer the diameter is greater than any other line CE which is more remote. If the given lines CD and CE be at the same side of CB, draw AD and AE, and in the triangles CAD, CAE, the sides CA and AD are equal to the sides CA and AE, and the angle CAD is greater than the angle CAE, and therefore the remaining side CD is greater than the remaining side CE (by Prop. 24. B. 1). But if the given lines CL and CE be at different sides of CB, construct the angle ACD equal to ACL, and CD will be equal to CL (by Part 3), but CD is greater than CE, and therefore CL is greater than CE.

PART 5.—More than two equal lines cannot be drawn. For let any three right lines be drawn from the point C to the circumference, and any one of them may be part of the diameter, and therefore greater or less than either of the remainders (by Part first and second), or two of them will be at the same side of the diameter, and therefore unequal (by Part 4).

PROPOSITION VIII. THEOREM.

1st.—If there be any point assumed without a circle, and right lines be drawn from it to the circumference, those making equal angles with the line passing through the centre, will be equal.

2ndly.—And of those incident on the concave circumference, the greatest is that which passes through the centre.

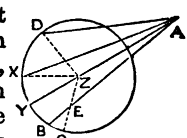
3rdly.—Of the remaining lines, that which is nearer to the line passing through the centre, is greater than the line which is more remote from the line passing through the centre.

4thly.—But of those incident on the convex circumference, that line is the least, which would pass through the centre if produced.

5thly.—Of the remaining lines, that, which is nearer to the least, is less than the line which is more remote from the least.

6thly.—Only two equal lines can be drawn, either to the concave or convex circumference.

PART 1.—AB and AX which make equal angles with AZ, are equal.



But if not, if it be possible, let either of them AB be greater than the other, and make AE equal to AX, and draw ZE and ZX, therefore in the triangles ZAE, ZAX the side ZA is common to both, and AE is equal to AX (by Constr.), the angle ZAE is also equal to ZAX (by Hypoth.), and therefore the sides ZE and

ZX are equal (by Prop. 4. B. 1), but the right line ZO is equal to ZX (by Def. 15. B. 1), and therefore ZE is equal to ZO; a part equal to the whole, which is absurd.

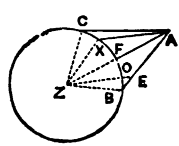
Therefore neither AB nor AX is greater than the other, and therefore they are equal.

PART 2.—Of those incident on the concave circumference, that line AY which passes through the centre is greater than any other AX. Draw ZX, and ZY will be equal to ZX (by Def. 15. B. 1), and therefore if the common line AZ be added to both, AY will be equal to AZ and ZX taken together, but AZ and ZX together are greater than AX (by Prop. 20. B. 1), and therefore AY is greater than AX.

PART 3.—AB or AX which is nearer the greatest, is greater than AD which is more remote.

If the given lines AX and AD be at the same side of AY, draw ZX and ZD, and in the triangles AZX, AZD, the sides AZ, ZX are equal to the sides AZ, ZD, and the angle AZX is greater than AZD, and therefore the remaining side AX is greater than the remaining side AD (by Prop. 24. B. 1). But if the given lines AB and AD be at different sides of AY, construct the angle ZAX equal to ZAB, and AX will be equal to AB (by Part first), but AX is greater than AD, and therefore AB is greater than AD.

PART 4.—Of those falling (or incident) on the convex circumference, that line AF, which if produced would pass through the centre, is less than any other line AX. For draw ZF and ZX, and ZX and XA are greater than ZA (by Prop. 20. B. 1), and therefore if the equals ZX and ZF be taken away, AX will be greater than AF.



PART. 5.—AB or AX which is nearer the least line, is less than AC which is more remote.

If the given lines AX and AC be at the same side of AZ, draw ZX and ZC; and ZC and CA taken together are greater than ZX and XA together (by Prop. 21. B. 1), therefore take away the equals ZC and ZX, and AC will be greater than AX.

But if the given lines AB and AC be at different sides of AZ, construct the angle ZAX equal to the angle ZAB, and AX will be equal to AB (by Part first), but AC is greater than AX, and therefore greater than AB.

PART 6.—Only two equal right lines can be drawn either to the concave or convex circumference.

For if any three lines be drawn, either one of them shall pass through the centre, and therefore be either greater or less than the others (by Parts 2 and 4), or two of them will be at the same side of the line passing through the centre, and therefore unequal (by Parts 3 and 5).

SCHOL.—Hence it appears easily, that any right line drawn to the concave circumference, is greater than a right line drawn to the convex; it follows that, if any three right lines are drawn from a point without a circle to its circumference, only two of them can be equal.

COR.—Hence, and from Part 5, Prop. 7, it appears, that there is no other point, except its centre, from which three equal right lines can be drawn to the circumference.

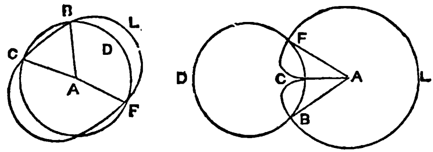
PROPOSITION IX. THEOREM.

If any point be assumed within a circle, from which to the circumference more than two equal right lines can be drawn, that point shall be the centre of the circle.

For it is not different from the centre, as if it were, only two right lines could be drawn from it to the circumference (by Prop. 7. B. 3).

PROPOSITION X. THEOREM.

A Circle cannot cut another in more than two points.



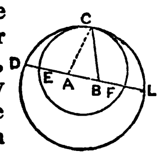
For let them cut (if it be possible) in three points, B, F and C, and let A be the centre of the circle BLF, and draw from it to the points of intersection the right lines AB, AF and AC, and these shall be equal (by Def. 15. B. 1); but because the circles intersect they have not the same centre (by Prop. 5. B. 3), therefore A is not the centre of the circle BDF, and therefore because three right lines AB, AC, and AF, are drawn to the circumference of the circle BDF, from a point different from the centre, these right lines will not be equal (by Cor. Prop. 8. B. 3), but they have been shown to be equal, which is absurd: the circles therefore do not intersect in three points.

SCHOL.—Hence it appears that a circle cannot meet another circle in more than two points.

PROPOSITION XI. THEOREM.

If two circles (ECF and DCL) touch each other internally, the right line joining their centres will pass through the point of contact.

But if not, if it be possible, let A be the centre of the circle ECF, and B the centre of the circle DCL, and let DL be the right line joining the centres; and draw BC and AC.

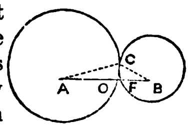


Therefore, because in the triangle BAC, the sides BA and AC are greater than the side BC (by Prop. 20. B. 1), and BC is equal to BD, because they are the radii of the circle DCL, the lines BA and AC will be greater than

BD; and therefore take away the common part BA, and AC will be greater than AD; but AC is equal to AE. because they are the radii of the circle ECF, and therefore AE is greater than AD, a part than the whole, which is absurd. Therefore the centres are not so placed, that a right line joining them cannot pass through the point of contact.

PROPOSITION XII. THEOREM.

If two circles (AOC and BFC) touch one another externally, the light line joining their centres passes through the point of contact.

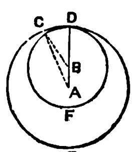


But if not, if it be possible, let A and B be the centres, and let the right line AB joining them, not pass through a point of contact, and draw AC and BC. Therefore, because in the triangle ACB, the sides AC and CB are greater than the side AB (by Prop. 20. B. 1), and the right line AO is equal to AC, because they are the radii of the circle AOC, and the right line BF is equal to BC, because they are the radii of the circle BFC; AO and BF together will be greater than BA, a part than the whole, which is absurd.

Therefore the centres are not so placed, that a right line joining them cannot pass through the point of contact.

PROPOSITION XIII. THEOREM.

One circle can touch another, either internally or externally, in only one point.

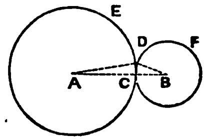


For if it be possible, let the circles ADE and BDF touch internally in two points, D and C, and draw AB joining the centres of the circles, and produce it until it passes through either of the points D, and draw AC and BC.

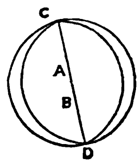
Therefore, because BD and BC are the radii of the circle BDF, BD is equal to BC; and therefore, if AB be added to both, AD will be equal to AB and BC together; but AD and AC are the radii of

the circle ADE, and therefore, AD is equal to AC, and therefore, AB and BC are equal to AC, but they are greater, (by Prop. 20. B. 1,) which is absurd.

Now if it be possible, let the two circles ADE and BDF touch in two points D and C externally, and draw the right line AB, joining the centres of the circles, and passing through either of the points of contact C, and draw AD and DB. Therefore, because AD and AC are the radii of ADE, they are equal; and because BC and BD are the radii of the circle BDF, they are also equal; therefore, AD and BD together are equal to AB, but they are greater than it, (by Prop. 20. B. 1), which is absurd.



But if the points of contact be at the extremities of the line joining the centres, CD must be bisected in A, and also in B, because it is the diameter of both circles, which is absurd.



Therefore, there is no case in which two circles can touch each other in two points.

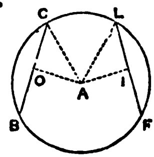
SCHOL.—Hence it appears that a circle touching another circle cannot meet it again.

PROPOSITION XIV. THEOREM.

PART 1.—In a circle, equal right lines, (BC and FL,) are equally distant from the centre.

PART 2.—And right lines, (BC and FL,) which are equally distant from the centre are equal.

PART 1.—Let A be the centre of the circle, and draw AC and AL, and also AO and AI, perpendicular to BC and FL. Therefore because BC and FL are equal, (by Hypoth.), and perpendiculars AO and AI drawn from the centre bisect them, (by Prop. 3. B. 3), OC and IL, will be also equal, (by Ax. 7,) and therefore, the squares of them are equal, (by Cor. 1, Prop



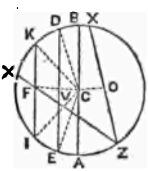
46. B. 1); but AC and AL are equal, and therefore the squares of them are equal, (by Cor. 1, Prop. 46. B. 1); but the square of AC is equal to the squares of AO and OC, (by Prop. 47. B. 1), and the square of AL is equal to the squares of AI and IL, (by Prop. 47. B. 1), and therefore, the squares of AO and OC, are equal to the squares of AI and IL; and therefore, take away the equal squares of OC and IL, and the squares of AO and AI will be equal, (by Ax. 3), and therefore, the right lines themselves are equal (by Cor. 2. Prop. 46. B. 1).

PART 2.—For since AO and AI are equal, (by Hypoth.), the squares of them are equal, (by Cor. 1, Prop. 46. B. 1); but AC and AL are equal, and therefore the squares of them are also equal (by Cor. 1. Prop. 46. B. 1), but the square of AC is equal to the squares of AO and OC, (by Prop. 47. B. 1); and the square of AL is equal to the squares of AI and IL, (by Prop. 47. B. 1), and therefore the squares of AO and OC, are equal to the squares of AI and IL, and therefore take away the equal squares of AO and AI, and the squares of OC and IL will be equal, (by Ax. 3), and therefore the right lines themselves are equal, (by Cor. 2. Prop. 46. B. 1); but because AO and AI bisect BC and FL, (by Prop 3. B. 3), OC and IL are the halves of BC and FL, and since they are equal, the lines BC and FL are also equal, (by Ax. 6.)

PROPOSITION XV. THEOREM.

PART 1.—Of right lines inscribed in a circle, the greatest is the diameter.

PART 2.—And of the others, that which is nearer the centre is greater than the line more remote.



PART 1.—The diameter AB is greater than any other line ED. For draw CD and CE; and CD is equal to CB, and CE to CA (by Def. 15. B. 1), therefore AB is equal to CD and CE together; but CD and CE together are greater than ED (by Prop. 20. B. 1), and therefore AB is greater than ED.

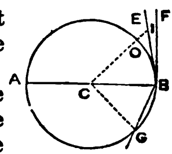
PART 2.—That which is nearer the diameter is greater than one more remote. First, let the given lines be ED and IK, which are at the same side, and do not intersect, and draw CD, CE, CI, and CK. Therefore because in the triangles ECD, ICK, the sides EC, CD, are equal to the sides IC, CK, and the angle ECD greater than the angle ICK, the side CD will be greater than the side IK (by Prop. 24. B. 1). Now, let the given lines be ZX and IK, which either are at different sides or intersect, and draw CO and CF perpendicular to ZX and IK, from the greater CF cut off CV equal to the less CO (by Prop. 3. B. 1), and through V draw ED perpendicular to CF.

Therefore because ZX and ED are equally distant from the centre (by Const.), ED is equal to ZX (by Prop. 14. B. 3), but ED is greater than IK (by Part 1), and therefore ZX is greater than IK.

PROPOSITION XVI. THEOREM.

PART 1.—The right line which is drawn from the extremity of the diameter of a circle perpendicular to it, wholly falls without the circle.

PART 2.—And if any right line be drawn from a point within it and the circle, to the point of contact, it cuts the circle.



PART 1.—For if it be possible, let BG meeting the circle again in G, be perpendicular to AB, and draw CG.

Therefore because in the triangle GCB, the side GC is equal to the side CB, the angle CBG is equal to the angle CGB (by Prop. 5. B. 1), and therefore each is an acute angle (by Cor. 1. Prop. 17. B. 1), but CBG is a right angle (by Hypoth.), which is absurd.

Therefore the right line drawn through B, perpendicular to AB, will not meet the circle again.

PART 2.—Let BF be perpendicular to AB, and from a point within it and the circle, draw to the point of contact, a right line, EB, which, if it be possible, does not cut the circle.

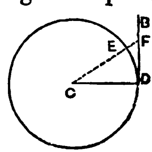
Therefore, because the angle CBF is right, CBE will be acute, therefore, draw CI perpendicular to BE, and it falls at the side of the angle CBE, (by Cor. 1. Prop. 16. B. 1.) Therefore in the triangle BCI, the angle CIB will be greater than CBI, and therefore the side CB will be greater than CI, (by Prop. 19. B. 1), but CB is equal to CO, and therefore CO will be greater than CI, a part greater than the whole, which is absurd. Therefore the line BI does not fall outside the circle, and therefore, the right line BE cuts the circle.

SCHOL. 1.—Hence it appears that the right line BF touches the circle in B; and that the tangent can only meet the circle in one point, and from each point of a circumference there is only one tangent.

SCHOL. 2.—It also appears that the right line, which from B makes with AB an acute angle however great, must meet the circle again.

COR. 1.—From this proposition a method appears for drawing a tangent, from any given point B in the circumference of a circle, that is to say, draw from that point the diameter AB and from the extremity of it draw the perpendicular BF.

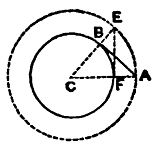
COR. 2.—If CD be produced beyond the extremity C, and there be taken in the produced part, any number of centres, and circles be described through the point D, it is evident that these circles touch DB in D, and cannot meet each other in any other point; whence the right line EF is cut by each circle in a different point, and therefore, *is infinitely divisible.*



PROPOSITION XVII.—PROBLEM.

From a given point (A), without a given circle (CBF, to draw a right line which shall touch the circle.

Let C be the centre of the given circle, and from the centre C, with the interval CA, describe the circle CAE; draw CA, and through F draw the right line FE perpendicular to CA, meeting the circle CAE in E, draw CE, and the right line drawn from the point B to the given point A touches the circle in B.



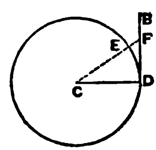
For in the triangles ACB, ECF, the sides AC, CB are equal to the sides EC, CF (by Def. 15. B. 1), and the angle at C is common to both, therefore the angle ABC is equal to the angle EFC (by Prop. 4. B. 1), but EFC is a right angle (by Constr.), and therefore ABC is a right angle, and therefore the right line AB touches the circle CFB (by Prop. 16. B. 3.)

SCHOL.—It appears that there can be two tangents drawn from the point A, that is to say, one from either side of the right line AC.

PROPOSITION XVIII. THEOREM.

If a right line (DB) touches a circle, the right line (CD) drawn from the centre to the point of contact, shall be perpendicular to it.

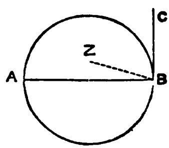
But if not, if it be possible, let CF be perpendicular to BD; and in the triangle CFD, because the angle CFD is a right angle, CDF will be acute (by Cor. Prop. 17. B. 1), therefore the side CD is greater than CF (by Prop. 19. B. 1), but CE is equal to CD, and therefore CE is greater than CF, a part than the whole, which is absurd.



Therefore CF is not perpendicular to BD, and it can similarly be proved that no other line is perpendicular to it, except CD.

PROPOSITION XIX. THEOREM.

If a right line (BC) touch a circle, the right line (BA) drawn from the point of contact perpendicular to the touching line, passes through the centre of the circle.

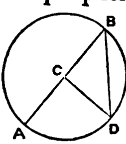


But if not, if it be possible, let the centre Z be without the right line BA, and draw ZB.

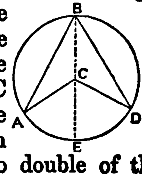
Therefore because the right line ZB is drawn from the centre to the point of contact, it is perpendicular to the touching line (by Prop. 18. B. 3); therefore ZBC is a right angle, but ABC is a right angle (by Hypoth.), therefore ZBC is equal to ABC, a part to the whole, which is absurd, therefore Z is not the centre, and it can similarly be shown that no other point without the right line AB, is the centre.

PROPOSITION XX. THEOREM.

The angle (ACD) at the centre of a circle, is double of the angle at the circumference, when they have the same part of the circumference for their base.

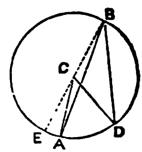


Firstly—Let either leg of the angle at the periphery pass through the centre; and because in the triangle DCB, the sides DC and BC are equal, the angles CBD and CDB are equal (by Prop. 5. B. 1), but the external angle ACD is equal to both together (by Prop. 32. B. 1), and therefore is double of ABD.



Secondly—Let the angle ACD fall within the angle ABD, and draw BCE; and because the angle ACE is equal to double of the angle ABC (by Part first), and the angle DCE is double of the angle DBC (by Part first), the angle ACD will be double of the angle ABC, together with double of DBC, and therefore equal to double of the angle ABD.

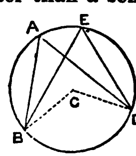
Thirdly—Let either leg of the angle ACD cut a leg of the angle ABD, and draw BCE, and because the angle ECD is double of the angle EBD (by Part first), or double the angle EBA, together with double the angle ABD, but the angle ECA is equal to double of the angle EBA (by Part first), take away these equal quantities from both, and the angle ACD shall be equal to double of the angle ABD.



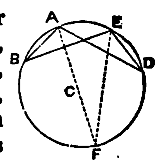
PROPOSITION XXI. THEOREM.

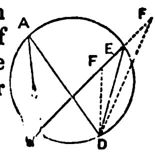
The angles (BAD and BED) in the same segment of a circle are equal to one another.

First—Let the segment BAD be greater than a semicircle, and let C be the centre of the circle, and draw CB and CD. Therefore because BCD at the centre, is double of either BAD or BED at the circumference (by Prop. 20. B. 3), BAD and BED will be equal (by Ax. 7).



Secondly—Let the segment be either a semicircle or less than a semicircle, and let C be the centre of the circle, and draw ACF and EF. Therefore, because the segment BDF is greater than a semicircle, and in it are the angles BAF and BEF, the angle BAF will be equal to BEF, (by First Part); and because the segment DEABF, is greater than a semicircle, and in it are the angles FAD and FED, the angle FAD will be equal to FED, (by First Part); therefore, the sum of the two BAF and FAD, or the angle BAD, is equal to the two angles BEF and FED, or to the angle BED.



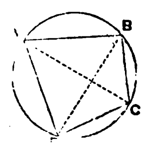


COR.—If two equal angles stand on the same arch BD, and the vertex of either of them BAD, be in the opposite circumference, the vertex of the other will be also in the same circumference.

But, if not, if it be possible, let it fall either outside or inside the circumference, as at F and draw FD; and because the angles BAD and BED are in the same segment, BAD will be equal to BED, (by Prop. 21. B. 3): but BAD is equal BFD (by Hypoth.) and therefore BED and BFD are equal; but one of them is greater than the other, (by Prop. 16. B. 1), which is absurd; therefore the point F does not fall within or without the circle, therefore, it falls on the circumference.

PROPOSITION XXII. THEOREM.

The opposite angles of a quadrilateral figure, (FABC) inscribed in a circle, are equal to two right angles.

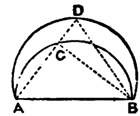


For, draw AC and FB, and because the angles ACB and AFB are in the same segment AFCB, ACB is equal to AFB (by Prop. 21. B. 3); and because the angles ACF and ABF are in the same segment ABCF, ABF is equal to ACF, (by Prop. 21. B. 3); therefore the whole angle BCF is equal to the two angles AFB and ABF together; but AFB and ABF together with FAB are equal to two right angles, (by Prop. 32. B. 1), and therefore, BCF together with FAB, is equal to two right angles; it can be similarly demonstrated that ABC and AFC are equal to two right angles.

COR.—If any side of a quadrilateral figure inscribed in a circle, be produced, the external angle will be equal to the internal remote angle, for each of them together with the internal adjacent are equal to two right angles, (by Prop. 13. B. 1, and Prop. 22. B. 3).

PROPOSITION XXIII. THEOREM.

Upon the same right line and upon the same part of it, two similar segments of circles cannot be constructed which do not mutually coincide.

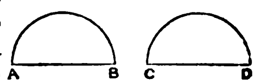


For, if it be possible, let ACB and ADB be constructed, and let any point D, in one of them, fall without the other, and draw DA, DB, and CB.

Therefore, because the segments ACB and ADB are similar, and in them are the angles ACB and ADB, ACB is equal to ADB, (by Def. 10. B. 3), but ACB is external to ADB, and therefore greater than it, (by Prop. 16. B. 1), which is absurd; therefore, no point in either of the segments falls without the other, therefore, they coincide.

PROPOSITION XXIV. THEOREM.

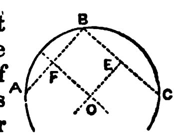
Similar segments of circles, standing upon equal right lines are equal.



For, if the equal right lines AB and CD, be so applied to each other, that the point A may fall on C, the point B must also fall on D, and therefore the right lines will coincide, (by AX. 10), and the segments will also coincide, (by Prop. 23. B. 3), and are therefore equal.

PROPOSITION XXV. PROBLEM.

A segment, (ABC) of a circle being given, to describe a circle of which it is a segment.

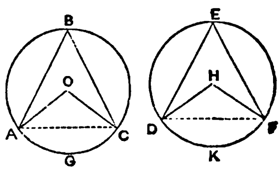


Let two right lines be drawn, not parallel, AB and CB, let them be bisected, and through the points of bisection F and E, let two right lines FO and EO, be drawn perpendicular to AB and CB; the intersection O of

these lines is the centre. Because the right line AB terminated in the circle, is bisected by the perpendicular FO, FO passes through the centre, (by Prop. 1. B. 3), EO also similarly passes through the centre, and therefore the centre is in O, the intersection of the lines FO and EO.

PROPOSITION XXVI. THEOREM.

In equal circles, (ABC, DEF,) equal angles, (AOC, and DHF, ABC and DEF,) whether they be at the centre or circumference, stand on equal arches.

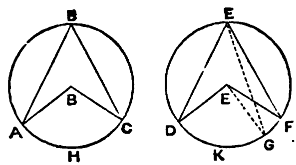


If the given angles AOC and DHF, be at the centre, let the angles ABC and DEF, at the circumference stand upon the same arch, and draw AC and DF. Therefore, because in the triangles AOC and DHF, the angles O and H are equal (by Hypoth.), and the sides AO and OC, equal to the sides DH and HF, (by Hypoth.), the bases AC and DF, are also equal; but the angles ABC and DEF are equal, (by Prop. 20. B. 3. and Ax. 7), and therefore the segments ABC and DEF, are similar, (by Def. 10. B. 3), but they stand on equal right lines, AC and DF, and therefore are equal, (by Prop. 24. B. 3); therefore, take away these equals from the equal circles, and the arches AGC and DKF remain equal.

It can be similarly demonstrated that the circles AGC and DKF, are equal, if the given angles ABC and DEF, at the circumference be acute, by drawing OA and OC, and also HD and HF. But if the given angles at the circumference be either right, or greater than right angles, bisect them, and the halves of them will be equal, and it can be shown, as above, that the arches upon which these halves stand, are equal, and therefore, the whole arches are equal.

PROPOSITION XXVII. THEOREM.

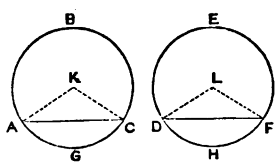
In equal circles, (ABC and DEF,) the angles (ABC, DEF), which stand upon equal arches, are equal, whether they be at the centre or circumference.



But, if not, if it be possible, let either of them, DEF, be greater than the other; and make the angle DEG equal to ABC. Therefore, because in the equal circles, ABC and DEF, the angle ABC is equal to the angle DEG, (by Const.) the arches AHC and DKG will be equal, (by Prop. 26. B. 3); but AHC and DKF are also equal (by Hypoth.), and therefore DKG is equal to DKF, a part to the whole, which is absurd; therefore, neither angle is greater than the other, and therefore, they are equal.

PROPOSITION XXVIII. THEOREM.

In equal circles (ABC, DEF), equal right lines (AC and DF) cut off equal arches, the greater equal to the greater (ABC to DEF), and the less to the less (AGC to DHF).

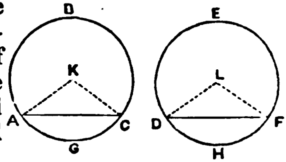


If the equal right lines be diameters, the proposition is evident. But if not, let K and L be the centres of the circles, and draw KA, KC, LD, and LF. Therefore, because the circles are equal (by Hypoth.), AK and KC will be equal to DL and LF; but AC and DF are equal (by Hypoth.), and therefore the angle AKC is equal to the angle DLF (by Prop. 8. B. 1): therefore the arch AGC is equal to the arch DHF (by Prop. 26. B. 3),

and since the circles are also equal, these equals being taken from them, the arches ABC and DEF will be equal.

PROPOSITION XXIX. THEOREM.

In equal circles (ABC, DEF) the right lines (AC and DF) which subtend equal arches, are also equal.



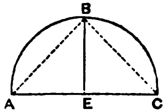
If the equal arches be semicircles, the proposition is manifest. But if not, let K and L be the centres of the circles, and draw AK, KC, DL and LF.

Therefore since the arches AGC and DHF are equal (by Hypoth.), the angles AKC and DLF will be equal (by Prop. 27. B. 3); but in the triangles AKC and DLF the sides AK, KC are also equal to the sides DL, LF (by Hypoth. and Def. 1. B. 3), and therefore the bases AC and DF are equal (by Prop. 4. B. 1).

SCHOL.—Whatever has been demonstrated in the four preceding propositions of equal circles, is also true of the same circle.

PROPOSITION XXX. PROBLEM.

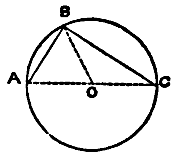
To cut an arch (ABC) into two equal parts.



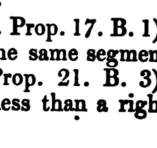
Draw AC, and bisect it in E, through E draw EB perpendicular to AC, and it will bisect the arch in B. For draw AB and CB, and in the triangles AEB, CEB, the sides AE and EC will be equal (by Constr.), and BE is common to both, and the angle AEB is equal to the angle CEB (by Constr.), therefore the sides AB and BC are equal (by Prop. 4. B. 1), and therefore the arches which they subtend are equal (by Prop. 28. B. 3), and therefore the given arch is bisected in B.

PROPOSITION XXXI. THEOREM.

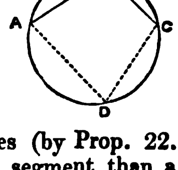
In a circle, the angle which is in a semicircle, is a right angle; but that in a greater segment is less than a right angle, and that in a less segment, is greater than a right angle.



PART 1.—The angle in the semicircle ABC is a right angle. For let O be the centre of the circle, and draw OB and AC. Therefore since in the triangle AOB, the sides OB and OA are equal, the angles OAB and OBA will be also equal (by Prop. 5. B. 1); OCB and OBC are similarly equal; therefore the angle ABC is equal to the two angles BAC and BCA taken together, and therefore the angle ABC is a right angle (by Cor. 1. Prop. 32. B. 1).



PART 2.—The angle ABC in a greater segment is less than a right angle. For draw AD a diameter of the circle, and let CD and CA be also drawn. Therefore since in the triangle ACD, the angle ACD in a semicircle is a right angle (by Part first), the angle ADC will be less than a right angle (by Cor. Prop. 17. B. 1), but the angles ADC and ABC are in the same segment ABDC, and are therefore equal (by Prop. 21. B. 3), and therefore the angle ABC is also less than a right angle.



PART 3.—The angle ABC in a segment less than a semicircle is greater than a right angle. For assume in the opposite circumference any point D, and draw DA and DC.

Therefore since in the quadrilateral figure ABCD, the two opposite angles B and D are equal to two right angles (by Prop. 22. B. 3); but the angle D, in a greater segment than a

semicircle, is less than a right angle (by Part second), therefore the angle B will be greater than a right angle.

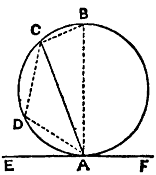
Cor. 1. (See first figure).—Hence a means appears of drawing a perpendicular from the extremity B of any given right line BC; for take any point O outside the given line, describe a circle passing through B, and cutting the given right line in any point C; and draw AC passing through O, draw AB, and this will be perpendicular to the given line CB; for ABC is an angle in a semicircle.

Cor. 2.—Hence also can be drawn from a given point outside a given circle, a right line touching it: draw a right line from the given point to the centre of the circle, and bisect it, and from the point of bisection describe a circle through the given point; a right line drawn from either intersection of the circle with the given circle will be touching the circle; for it is perpendicular to the radius, drawn to its meeting point with the circle, because the angle in a semicircle is a right angle.

PROPOSITION XXXII. THEOREM.

If any right line (EF) touches a circle, and from the point of contact a right line (AC) be drawn, cutting the circle, the angle (EAC) contained by the touching and cutting lines, will be equal to the angle (ABC) in the alternate segment of the circle.

Part 1.—If the cutting line passes through the centre, it is manifest that they are equal, for they are right angles (by Props. 18 and 31. B. 3).



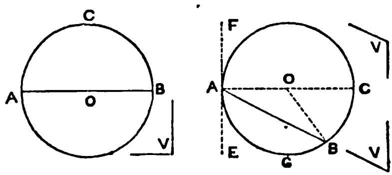
But if not, through the point of contact A draw AB perpendicular to EF, and draw BC. Therefore, since the right line EF touches the circle, and AB is drawn through the point of contact, perpendicular to EF, it passes through the centre (by Prop. 19. B. 3); and therefore the angle ACB is a right

angle (by Prop. 31. B. 3); therefore in the triangle ABC, the remaining two ABC and BAC together are equal to a right angle (by Cor. Prop. 32. B. 1), and therefore are equal to BAE; therefore take away the common angle BAC, and the angle CAE will be equal to the angle ABC, in the alternate segment.

PART 2.—And because in the quadrilateral ABCD, the angles ABC and ADC together are equal to two right angles (by Prop. 22. B. 3), EAC and FAC together will be equal to ABC and ADC (by Prop. 13. B. 1), and therefore, taking away the equals EAC and ABC (by Part First), FAC will be equal to the angle ADC, in the alternate segment.

PROPOSITION XXXIII. PROBLEM.

Upon a given right line (AB) to describe a segment of a circle, that shall have an angle equal to a given angle (V).



First let the given angle V be a right angle (in first figure), and bisect the given line in O; from the centre O, through A, describe a circle; and since it is divided, by the given right line, into segments which are semicircles, it is manifest that each of them contains an angle equal to the given right angle V (by Prop. 31. B. 3).

But let the given angle V be acute or otherwise (as in second figure), make with the given line AB, from either extremity of it A, an angle EAB or FAB, equal to the given angle V.

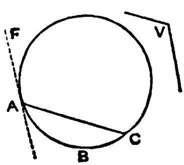
And through A draw AC perpendicular to EF, and at B make an angle ABO equal to BAC. The circle described from the centre O, with the interval OA, will

pass through B, because OA and OB are equal (by Const. and Prop. 6. B. 1); and the segment ACB will contain an angle equal to the given acute angle V; but the segment AGB will contain an angle equal to the given obtuse angle V. Because the right line EF touches the circle in A (by Const. and Prop. 16. B. 3), and from A is drawn AB, which cuts the circle, the angle in the segment ACB is equal to the angle EAB (by Prop. 32. B. 3), and therefore is equal to the given acute angle V (by Const.); and also the angle in the segment AGB is equal to the angle FAB (by Prop. 32. B. 3), and therefore is equal to the given obtuse angle V (by Const.)

SCHOL.—A circle can be similarly described, which touches a given right line EF, and passes through a given point without that line.

PROPOSITION XXXIV. PROBLEM.

From a given circle (ABC) to cut off a segment which will have an angle equal to a given angle (V).



Draw FA touching the circle in any point A; and at the point A, with the right line AF, make an angle FAC equal to the given angle V; the segment ABC will contain an angle equal to the given angle V.

For since FA touches the circle, and AC cuts it, the angle in the segment ABC will be equal to the angle FAC (by Prop. 32. B. 3), and therefore equal to the given angle V (by Const.).

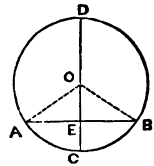
PROPOSITION XXXV. THEOREM.

If two right lines (AB and CD) within a circle cut each other, the rectangle under the segments (AE and EB) of one of them is equal to the rectangle under the segments (CE and ED) of the other.

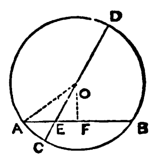
CASE 1.—If both of them pass through the centre,

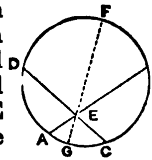
they are bisected in the centre, and therefore the rectangles under these equal segments are the squares of the halves, and therefore are equal, which is evident.

Case 2.—If one of them DC pass through the centre, and the other not, but cuts DC perpendicularly. Draw OA and OB; and because DC is cut equally in O, and unequally in E, the rectangle under DE, and EC, together with the square of OE, will be equal to the square of OC (by Prop. 5. B. 2), or the square OB; and therefore to the squares of OE and EB (by Prop. 47. B. 1); take away the common square of OE, and the rectangle under DE and EC will be equal to the square of EB, or, because AE and EB are equal (by Prop. 3. B. 3), to the rectangle under AE and EB.



Case 3.—If one of them DC pass through the centre, and the other AB not passing through the centre, cuts DC obliquely, draw OF perpendicularly to AB, and also OA; and because DC is cut equally in O, and unequally in E, the rectangle under DE and EC, together with the square OE, or the squares of OF and FE (by Prop. 47. B. 1), will be equal to the square of OC (by Prop. 5. B. 2), or to the square of OA, and therefore to the squares of OF and FA (by Prop. 47. B. 1), take away the common square of OF, and the rectangle under DE and EC, together with the square of EF, will be equal to the square of FA; but since OF is perpendicular to AB, it bisects it (by Prop. 3. B. 3), and therefore the rectangle under BE and EA, together with the square of FE, is equal to the square of FA (by Prop. 5. B. 2), or to the rectangle under DE and EC, together with the square FE, take away the common square of FE, and the rectangle under BE and EA will be equal to the rectangle under DE and EC.

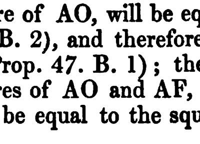




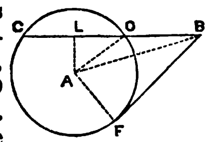
CASE 4.—But if neither of them pass through the centre, draw through their intersection a diameter FG, and the rectangle under FE and EG will be equal to the rectangle under DE and EC, and also to the rectangle under BE and EA (by Part First), and therefore the rectangle under DE and EC is equal to the rectangle under BE and EA (by Ax. 1).

PROPOSITION XXXVI. THEOREM.

If from a point (B) without a circle, two right lines be drawn to it, one of which (BC) cuts it, and the other (BF) touches it, the rectangle under the whole cutting line (BC), and the external segment (BO) will be equal to the square of the touching line.



CASE 1.—If BC pass through the centre, draw AF; and because the right line CO is bisected in A, and to the same right line, OB is added, the rectangle under the whole CB and the added part BO, together with the square of AO, will be equal to the square of AB (Prop. 6. B. 2), and therefore to the squares of AF and FB (by Prop. 47. B. 1); therefore taking away the equal squares of AO and AF, the rectangle under CB and BO will be equal to the square of BF (by Ax. 3).



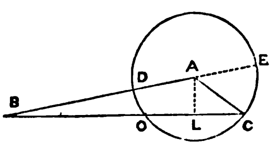
CASE 2.—If BC does not pass through the centre, draw AL perpendicular to it, and also draw AO, AB, and AF, and because CO is bisected in L (by Prop. 3. B. 3), and to the same right line OB is added, the rectangle under CB and BO, together with the square of LO will be equal to the square of

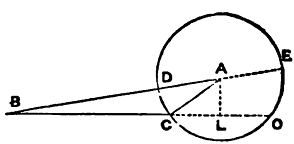
LB (by Prop. 6. B. 2); add to both the square of AL, and the rectangle under CB and BO, together with the squares of AL and LO, or the square of AO (by Prop. 47. B. 1), will be equal to the squares of AL and LB, and therefore to the square of AB (by Prop. 47. B. 1) or to the squares of AF and FB (by Prop. 47. B. 1); therefore taking away the equal squares AO and AF, the rectangle under CB and BO will be equal to the square of BF (by Ax. 3).

Cor. 1.—Hence if from any point without a circle, two right lines be drawn cutting the circle, the rectangles under them and their external segments will be equal, for each is equal to the square of a touching line.

Cor. 2.—If from the same point two right lines be drawn to a circle, which touch the circle, they will be equal; for their squares are equal, because either of them is equal to the same rectangle; that is, the rectangle under any line drawn from the same point, terminating on the concave of the circle, and its external segment.

Cor. 3.—If from the angle A of a triangle BAC a perpendicular AL be let fall on the opposite side, the rectangle under the sum BE and difference BD of the sides AB and AC, will be equal to the rectangle under the sum and difference of the segments BL and LC, intercepted between the perpendicular and the extremities of the side on which it falls.



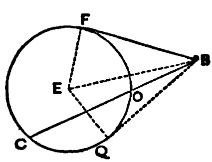


For from the centre A, and with the less side AC as an interval, describe a circle; and since the right line

BE and BC cut the circle, the rectangle under EB and DB will be equal to the rectangle under CB and BO (by Cor. 1), but AE is equal to AC, and therefore EB is equal to the sum of AC and AB; but AD is also equal to AC, and therefore BD is equal to the difference between AC and AB; and therefore the rectangle under EB and DB is the rectangle under the sum and difference of AB and AC; but because AL is perpendicular to CO, OL and LC are equal (by Prop. 3. B. 3); and therefore when the perpendicular falls within the triangle, BO is equal to the difference of BL and LC, but BC is their sum; and when the perpendicular falls outside the triangle (as in the other figure), BO is equal to the sum of them, and BC the difference, and therefore the rectangle under BC and BO is the rectangle under the sum and difference of them, BC and BO; and is equal to the rectangle under the sum and difference of the sides AB and AC.

PROPOSITION XXXVII. THEOREM.

If from a point (B) without a circle two right lines be drawn to it, either of them (BC) cutting it and the other (BF) meeting it, and if the rectangle under the cutting line and its external segment, be equal to the square of the line which meets the circle, the meeting line (BF) touches the circle, or is a tangent to it.



For draw from B the right line BQ touching the circle, and also draw EF and EQ. Therefore since the right line BC cuts the circle, and BQ touches it, the square of BQ will be equal to the rectangle under BC and BO (by Prop. 36. B. 3); but the square of BF is equal to the rectangle under BC and BO (by Hypoth.), and therefore the squares of BQ and BF are equal, and therefore the right lines themselves are equal (by Cor.

2. Prop. 46. B. 1); now in the triangles EFB and EQB, the sides EF and FB, are equal to the sides EQ and QB, but the side EB is common, and therefore the angle EFB is equal to EQB (by Prop. 8. B. 1); but the angle EQB is a right angle (by Prop. 18. B. 3), and therefore the angle EFB is right, and therefore the right line BF touches the circle (by Prop. 10. B. 3), and is therefore a tangent to it.

QUESTIONS

REFERRING TO THE THIRD BOOK.

How do you complete a circle, having only a segment of it given?

Prove the 22nd Proposition, by assuming numbers for each angle?

Upon what previous Propositions does the proof of the 20th Proposition depend?

By what Proposition can you erect a perpendicular at the extremity of a given line?

MULTIPLICATION OF ALGEBRA.

RULE.

First.—When the quantities (or letters) are similar; multiply the numerical co-efficients, as in Arithmetic; then add together the indices of the similar quantities, placing their sum over the common letter or letters; and if the multiplier and multiplicand have like signs, place plus before the product, but if unlike, place minus before it.

Secondly.—When the quantities are not similar; multiply the numerical co-efficients, as before, and write down the unlike quantities after their product, prefixing, as above, the required sign to the result.

Ex 1.	Ex. 2.	Ex. 3.	Ex. 4.	Ex. 5.
$+a$	$-a^2$	a^2	$a^{\frac{1}{2}}$	$\sqrt{a}$
$+a$	$-a^4$	$-a^4$	$a^{\frac{1}{2}}$	$\sqrt{a}$
$+a^2$	$+a^6$	$-a^6$	a	a

In Ex. 1, we have $+a$ to be multiplied by $+a$, that is, the first power of a to be multiplied by itself, which is equal to $a \times a$, or aa, or a^2, joined by adding

the indices, according to the Rule; it must be remembered that a is the first power of a, therefore it has one for its index, thus a^1. Again, the multiplier and multiplicand having like signs, the product becomes plus. In Ex. 2, the product is similarly found, and the signs being similar to each other, produce plus, as in the last example. But in Ex. 3, the signs are not similar, therefore they make minus.

Exs. 4 and 5 are both the same, therefore the index of $\sqrt{a}$ being $\frac{1}{2}$, the sum of the indices becomes 1, that is a^1. Wherefore multiplication of similar quantities is performed by adding their indices, and prefixing the sum to the right hand of the common letter or letters.

Ex. 6.	Ex. 7.	Ex. 8.	Ex. 9.
a	a^2	a^2	$a^2 b^2$
b	$-b^2$	b^2	$a^2 b^2$
ab	$-a^2 b^2$	$a^2 b^2$	$a^4 b^4$

The product of each of the Examples 6, 7, and 8, is found by the second part of the general Rule; and the product of the 9th is found by both first and second part; namely, by first adding the indices of the similar quantities, and then writing them down as above. Let it now be remembered that a letter standing without a co-efficient, 1 is understood, therefore a means $1a$, and b, $1b$.

Ex. 10.	Ex. 11.	Ex. 12.
$2a - 3x$	$2a + 3x$	$2\sqrt{a} + 2\sqrt{x}$
$2a$	$2a$	$2a$
$4a^2 - 6ax$	$4a^2 + 6ax$	$4\sqrt{a^3} + 4ax^{\frac{1}{2}}$

Ex. 13.

$$\begin{array}{l} 5 - 2\sqrt{6} \\ 10 + 2\sqrt{6} \\ \hline 50 - 20\sqrt{6} \\ \quad\;\; 10\sqrt{6} - 24 \\ \hline 26 - 10\sqrt{6} \\ \hline \end{array}$$

Ex. 14.

$$\begin{array}{l} \sqrt{a} + \sqrt{b} \\ \;\; a + \quad b \\ \hline \sqrt{a^3} + a\sqrt{b} \\ \qquad\quad \sqrt{a}\,b + \sqrt{b^3} \\ \hline \sqrt{a^3} + a\sqrt{b} + \sqrt{ab} + \sqrt{b^3} \\ \hline \end{array}$$

These five Examples require but little explanation, as all relative to them will appear evident, by strict attention to the adding of the indices of similar quantities. In the 14th Example, the index of $\sqrt{a}$ is $\frac{1}{2}$, and the index of a is 1, therefore, by adding these indices, we have $1\frac{1}{2} = \frac{3}{2}$, for the index of the product of $\sqrt{a}$ multiplied by a; and then, multiplying $\sqrt{b}$ by a, we have $a\sqrt{b}$. Next, multiplying by the quantity b, the product will be $\sqrt{ab} + b^{\frac{3}{2}}$, and the signs of the multiplier and multiplicand being similar, each product requires an affirmative sign; therefore, by Addition, the whole result will be $a^{\frac{3}{2}} + ab^{\frac{1}{2}} + a^{\frac{1}{2}}\,b + b^{\frac{3}{2}}$, which is the same with $\sqrt{a^3} + a\sqrt{b} + \sqrt{ab} + \sqrt{b^3}$. Examples 12 and 13 are similarly worked.

Ex. 15.

$$\begin{array}{l} a^n + b^n \\ a^n - b^n \\ \hline a^{2n} + a^n b^n \\ \qquad\; - a^n b^n - b^{2n} \\ \hline a^{2n} \quad * \quad - b^{2n} \\ \hline \end{array}$$

Ex. 16.

$$\begin{array}{l} a + b \\ a - b \\ \hline a^2 + ab \\ \qquad - ab - b^2 \\ \hline a^2 \quad * \quad - b^2 \\ \hline \end{array}$$

By adding the indices in the 15th example, it is evident we have $2n$ for the index of a. In this, and the 16th example, we have the solution of the Geometrical Theorem, given as the 3rd Corollary of the 5th Proposition of the Second Book.

Ex. 17.

$$\begin{array}{l} a^{n-1} + b^{n-1} \\ a^{n-1} + b^{n-1} \\ \hline a^{2n-2} + a^{n-1}b^{n-1} \\ \qquad\qquad a^{n-1}b^{n-1} + b^{2n-2} \\ \hline a^{2n-2} + 2a^{n-1}b^{n-1} + b^{2n-2} \\ \hline \end{array}$$

Ex. 18.

$$\begin{array}{l} a + b \\ a + b \\ \hline a^2 + ab \\ \qquad ab + b^2 \\ \hline a^2 + 2ab + b^2 \\ \hline \end{array}$$

It is evident that $n - 1$ added to $n - 1$ becomes $2n - 2$, for the index of a; and the indices of the other products may be similarly found. In these two Examples we have the proof of the 4th Proposition of the Second Book, a^2 and b^2 being the squares of the parts, and $2ab$ equal to twice the rectangle under the parts.

Ex. 19.

$$
\begin{array}{l}
x^3 + ax^2 + b \\
x^3 - ax^2 + c \\
\hline
x^6 + ax^5 + x^3b \\
\quad - ax^5 - a^2x^4 - ax^2b \\
\qquad\qquad cx^3 + acx^2 + bc \\
\hline
x^6 + (b + c)x^3 - a^2x^4 - (ab - ac)x^2 + bc \\
\hline
\end{array}
$$

Ex. 20.

$$
\begin{array}{l}
(b - c)\, a^3 + (b^3 - b^2c + bc^2 - c^3)\, a^2 + 3a \\
b + c \\
\hline
(b^2 - cb)\, a^3 + (b^4 - b^3c + b^2c^2 - bc^3)\, a^2 + 3ab \\
\qquad (bc - c^2)a^3 + (b^3c - b^2c^2 + bc^3 - c^4)a^2 + 3ac \\
\hline
(b^2 - c^2)a^3 + (b^4 - c^4)a^2 + 3(c + b)a \\
\hline
\end{array}
$$

Here, as in Addition and Subtraction, we find the advantage of a parenthesis, to enclose all those quantities which are multiplied by the same co-efficient; it is evident that the work would otherwise be much longer, as the above Answers written at length would be the following:—

19th Answer....$x^6 + bx^3 + cx^3 - a^2x^4 - abx^2 + acx^2 + bc$.

20th Answer....$b^2a^3 - c^2a^3 + b^4a^2 - c^4a^2 + 3ac + 3ab$.

CASE 2.

EXAMPLES WITH ANSWERS.

1.—Multiply $a^{-y} + b^{-z}$ by $a^{-y} + b^{-2z}$

Ans... $a^{-2y} + a^{-y}b^{-z} + a^{-y}b^{-2z} + b^{-3z}$

2.—Multiply $a^3b + a^4 + a^yb^2 - a^4b + a^4b^2$ by $- a^3 - a^2b^2 - a^3b + a^2b$.

Ans...$(1 - b^2)b^2a^5 - (4 - b + b^2)b^2a^6 - (1 + b^3)a^7$.

3.—Multiply $3x^2y^2 - 2xy + 6$ by bxy.

Ans...$3bx^3y^3 - 2bx^2y^2 + 6bxy$.

4.—Multiply $x^2 - ax - b$ by $ax - b$.

Ans...$ax^3 - (a^2 + b)x^2 + b^2$.

5.—Multiply $ba^m - qa^n - ra^v$ by $ba^m - 2qa^n - 3ra^v$

Ans...$3bqa^{m+n} - 4bra^{m+v} + 5qra^{n+v} + b^2a^{2m} + 2q^2a^{2n} + 3r^2a^{2v}$

It will be found essential to bear in mind the nature of the indices, and then the student can advance with ease to himself. By referring to the 12th Definition, (given at the end of 1st Book) you will see the properties of the 1st Example in this Case: but it may be further added, that a^{-y} indicates the nought power of $1a$ divided by a^y thus, $\frac{a^0}{a^y}$. Again, $\frac{a^y}{a^y} = 1$; and $\frac{a^y}{a^y}$ indicates a^{y-y}, therefore since y subtracted from y leaves the nought power of a, it is evident that a^0 always indicates unity; therefore $a^{-y} = \frac{a^0}{a^y} = \frac{1}{a^y}$ Similarly $2a^{-y}$ means $\frac{2}{a^y}$, and 6×5^{-2} means $\frac{6}{5^2} = \frac{6}{25}$. We shall again comment on this topic, under the head of Roots.

DIVISION.

RULE I.

Arrange the quantities in proper order, as in Arithmetic; and, this Rule being the reverse of Multiplication, we subtract the indices of similar letters, and place the others in the quotient, as we find them in the dividend. Ex. gr.: $\frac{a^4}{a} = a^3$, because by subtracting the index of the divisor from that of the dividend, there will remain for the quotient, a^3; and $\frac{a^2d^3}{a} = ad^3$; it is also to be understood that numerical co-efficients are to be divided, as in Arithmetic, and that like signs produce an affirmative result, but unlike signs produce minus.

EXAMPLES WITH ANSWERS.

1.—Divide $12a^6x^8$ by $4a^2x^4$.

Ans...$3a^4x^4$.

2.—Divide $3a^2b^2$ by ab.

Ans...$3ab$.

3.—Divide $4a^7bd$ by $-2a^6bd$.

Ans... $-2a$.

4.—Divide $8abd$ by ac.

Ans...$8\frac{bd}{c}$.

It may be well to remark here that similar quantities in divisor and dividend, having equal powers, destroy each other, because their indices, being subtracted, become nought; and any letter in the divisor, not found in the dividend, must still be retained as a divisor, in a fractional form.

RULE II.

In compound quantities having a simple divisor, we proceed by dividing each quantity by it, prefixing the proper sign, according to the last Rule.

1.—Divide $24ab^3c - 10b^2cx^2 - 8abcy - 8bc$ by $2bc$.

$$2bc)\underline{24ab^3c - 10b^2cx^2 - 8abcy - 8bc}$$
$$12b^2a - 5bx^2 - 4ay - 4$$

2.—Divide $10ad - abx + ars + amn + ax^2 + xy$ by $-a$.

$$-a)\underline{10ad - abx + ars + amn + ax^2 + xy}$$
$$-10d + bx - rs - mn - x^2 - \frac{xy}{a}$$

It will be requisite to remember that any number of quantities requiring to be multiplied together, and divided by any one of them, the quotient is to be found by dividing by that quantity, and noting down the unlike quantities as found in the dividend; therefore, $10ad \div -a$ produces $-10d$. which is evident from the following numerical proof;—suppose $a = 4$, and $d = 6$, then $10.4.6 \div 4 = 10.1.6 = 60$; and if we first multiply the three numbers together, and then divide by the similar number, we shall have the same

result (Scilicet, $10.4.6 = 240 \div 4 = 60$); but with reference to the last quantities of the 2nd Example, it appears that it is not divisible by $-a$, not having a similar quantity contained in it, therefore it is necessary to write it in the form of a fraction, as $\frac{xy}{a}$.

RULE III.

When the divisor and dividend consist of compound quantities, it will be necessary to arrange them according to the index of a similar quantity in each, so that the highest power may be first: therefore having $a^2 + a^3b - a^4c + a^6d$ to be divided by $a + b$; and, arranging them according to the power or index of a, we place them thus, $a^6d - a^4c + a^3b + a^2$; and then proceed by dividing the first term of the dividend by the first term of the divisor, placing the result in the quotient. We next multiply the whole divisor by the quotient thus found, and subtract the result from the dividend, for a new dividend; bringing down other terms, when required, if any. Should there be a remainder not divisible by the divisor, put it in a fractional form, as in Arithmetic.

Ex. 1.

$$
\begin{array}{l}
a - c)a^2 - 2ac + c^2(a - c \\
\qquad\;\; a^2 - ac \\
\hline
\qquad\qquad\; ac + c^2 \\
\qquad\quad\; - ac + c^2 \\
\hline
\qquad\qquad\;\; * \quad\; * \\
\hline
\end{array}
$$

In this Example we divide a into the first quantity, and find a for the quotient, then multiplying $a - c$

by a, according to the Rule, the product will be $a^2 - ac$; next we change (or suppose changed) the signs of $a^2 - ac$, and subtract them from the dividend; the remainder will then be $- ac$, to which next bring down c^2, and proceed as before, and in all cases.

Ex. 2.

$$
\begin{array}{l}
a + b)a^4 - 3b^4(a^3 - a^2b + ab^2 - b^3 - \dfrac{2b^4}{a+b} \\
\quad a^4 + a^3b \\
\quad \overline{\qquad - a^3b - 3b^4} \\
\qquad - a^3b - a^2b^2 \\
\qquad \overline{\qquad\qquad a^2b^2 - 3b^4} \\
\qquad\qquad\quad a^2b^2 + ab^3 \\
\qquad\qquad\quad \overline{\qquad\quad - ab^3 - 3b^4} \\
\qquad\qquad\qquad\quad - ab^3 - \quad b^4 \\
\qquad\qquad\qquad\quad \overline{\qquad\qquad - 2b^4} \\
\qquad\qquad\qquad\qquad\qquad\quad \overline{a + b}
\end{array}
$$

Here we have a remainder, $- 2b^4$, under which the divisor is to be placed to form a fraction, because it is not divisible by it.

EXAMPLES WITH ANSWERS.

1.—Divide $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ by $a^2 - 2ab + b^2$.

Ans...$a^3 - 3a^2b + 3ab^2 - b^3$.

2.—Divide $6x^4 + 9x^2 - 20x$ by $3x^2 - 3x$.

Ans...$2x^2 + 2x + 5 - \dfrac{5x}{3x^2 - 3x}$

3.—Divide $9x^3 - 72$ by $3x - 6$.

Ans...$3x^2 + 6x + 12$.

4.—Divide $x^6 - x^4 + x^3 - x^2 + 2x - 1$ by $x^2 + x - 1$.

Ans...$x^4 - x^3 + x^2 \quad x + 1$.

5.—Divide $x^4 - a^2x^2 + 2a^3x - a^4$ by $x^2 - ax + a^2$.

Ans...$x^2 + ax - a^2$.

6.—Divide $9x^6 - 46x^5 + 95x^2 + 150x$ by $x^2 - 4x - 5$.

Ans...$9x^4 - 10x^3 + 5x^2 - 30x$.

The parenthesis is also requisite in Division, as well as the preceding Rules, and the student is requested to remember that all within such is considered as a single quantity. It will also be seen, by the following Examples, that literal indices are to be subtracted as other indices, placing the remainder for an index to their common letter or letters.

$$
\begin{array}{l}
b - a)b^3 - nab^2 + na^2b - a^3(b^2 - (n - 1)ab + a^2 \\
\quad\;\; b^3 - ab^2 \\
\hline
\qquad - (n - 1)\, ab^2 + na^2b \\
\qquad - (n - 1)\, ab^2 + (n - 1)\, a^2b \\
\hline
\qquad\qquad\qquad\qquad\qquad a^2b - a^3 \\
\qquad\qquad\qquad\qquad\qquad a^2b - a^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad\;\; * \quad * \\
\hline
\end{array}
$$

Having divided by the first term of the divisor, and having multiplied and subtracted as usual, we have for a remainder, $- nab^2 + ab^2$; but, as each quantity

has a similar co-efficient, it can be contracted by a parenthesis, as we have before shown, thus, $-(n - 1)ab^2$, $+ ab^2$ becoming $- 1$, because the sign preceding the parenthesis is minus. Next, $-(n - 1)$ is to be placed in the quotient and the first term of the divisor is then to be divided into its co-efficient, namely, into ab^2, placing the result also in the quotient, and proceed as usual, by multiplying $b - a$ by $-(n - 1)ab$; then subtract the product, and divide as before.

EXAMPLES WITH LITERAL INDICES.

1.—Divide $4x^n - 8xy^{n+1} - 8x^{n-1}y + 16y^{n+2}$ by $2x^{n-1} - 4y^{n+1}$

$$2x^{n-1} - 4y^{n+1})$$
$$4x^n - 8xy^{n+1} - 8x^{n-1}y + 16y^{n+2}\,(2x - 4y$$
$$\underline{4x^n - 8xy^{n+1}}$$
$$- 8x^{n-1}y + 16y^{n+2}$$
$$\underline{- 8x^{n-1}y + 16y^{n+2}}$$
$$* \quad * \quad *$$

2.—Divide $a^{2n} - 2a^nb^n + b^{2n}$ by $a^n - b^n$.

Ans...$a^n - b^n$.

3.—Divide $a^{2n-2} + 2a^{n-1}b^{n-1} + b^{2n-2}$ by $a^{n-1} + b^{n-1}$.

Ans...$a^{n-1} + b^{n-1}$.

4.—Divide a^n by a^{n-1}.

Ans...a.

5.—Divide a^{n-1} by a^{2n}.

Ans...a^{-n-1}.

EXAMPLES WITH RADICAL SIGNS.

1.—Divide $\sqrt{a^3} + a\sqrt{b} + \sqrt{ab} + \sqrt{b^3}$ by $\sqrt{a} + \sqrt{b}$.

$$\sqrt{a} + \sqrt{b})\sqrt{a^3} + a\sqrt{b} + \sqrt{ab} + \sqrt{b^3}(a + b$$
$$\underline{\sqrt{a^3} + a\sqrt{b}}$$
$$\sqrt{ab} + \sqrt{b^3}$$
$$\underline{\sqrt{ab} + \sqrt{b^3}}$$
$$* \quad *$$

By subtracting the index of $a^{\frac{1}{2}} = \sqrt{a}$, from the index of $a^{\frac{3}{2}} = \sqrt{a^3}$, we find the index of the first term for the quotient, that is, $\frac{1}{2}$ from $\frac{3}{2}$, $= \frac{2}{2} = 1$ remains for the index of a in the quotient; the rest is evident from what has been before explained relative to indices.

2.—Divide $a^{-\frac{5}{4}}$ by a.

$a = \sqrt[4]{a^4}$ and $a^{-\frac{5}{4}} = \frac{1}{\sqrt[4]{a^5}}$, then, by inverting the divisor, we have $\frac{1}{\sqrt[4]{a^4}} \times \frac{1}{\sqrt[4]{a^5}} = \frac{1}{\sqrt[4]{a^9}}$ for the Answer.

EXAMPLES FOR EXERCISE.

1.—Divide $x^5 - y^5$ by $x - y$.

Ans...$x^4 + x^3y + x^2y^2 + xy^3 + y^4$.

2.—Divide $x^3 - (a+p)x^2 + (ap+q)x - aq$ by $x^2 - px + q$.

Ans...$x - a$.

3.—Divide $a^{-5} + a^{-3}b^{-5} + a^{-2}b^{-4} + b^{-9}$ by $a^{-3} + b^{-4}$.

Ans...$a^{-2} + b^{-5}$.

4.—Divide $ab^3 - (a^2 + c)b^2 + c^2$ by $ab - c$.

Ans. $b^2 - ab - c$.

5.—Divide $(a^2 - 4)(a^2 - 4a)$ by $a^2 + 2a$.

Ans...$a^2 - 6a + 8$.

6.—Divide $a^{-6} + 2a^{-8} + a^{-10}$ by $a^{-3} + a^{-5}$

Ans...$a^{-3} + a^{-5}$

7.—Divide $a^{2n-2m} b^{-2n}$ by $a^{n-m} b^{-n}$

Ans...$a^{n-m} b^{-n}$

It is evident that these negative powers, as well as positive, are divided by subtracting the indices; the Answers can be also written as follow:—

Ex. 3. $a^{-2} + b^{-5} = \frac{1}{a^2} + \frac{1}{b^5}$

Ex. 6. $a^{-3} + a^{-5} = \frac{1}{a^3} + \frac{1}{a^5}$

Ex. 7. $a^{n-m} b^{-n} = \frac{a^n}{a^m} \times \frac{1}{b^n}$

ALGEBRAIC FRACTIONS.

The operations in Arithmetic and Algebraic Fractions being nearly similar, it is judged unnecessary to say much on the subject.

To reduce a mixed quantity to an improper fraction.

A mixed number is that which contains a whole number and a fraction, viz., $4 + \frac{5}{8}$ is a mixed number, and, in reducing it to an improper fraction, we must fine how many 8ths are in the whole number, thus, $4 \times 8 + 5 = \frac{32 + 5}{8}$ the improper fraction required.

RULE.

Multiply the whole number by the denominator of the fraction, and connect the numerator by the proper sign, placing the denominator under all.

1. $5x + \frac{2x - 3}{7} = \frac{35x + 2x - 3}{7}$

2. $a - \frac{b}{c} = \frac{ac - b}{c}$

3. $5a - \frac{3x - b}{a} = \frac{5a^2 - 3x + b}{a}$

4. $3ax - \frac{a - b}{y} - \frac{3axy \ldots a + b}{y}$

It will be here required to remember, that when a minus sign stands before a fraction with a compound numerator, each sign in that fraction must be changed.

To reduce an improper fraction to a whole or mixed number.

RULE.

Divide the numerator by the denominator, as in Arithmetic.

1. $\frac{2ay - b}{ay} = 2 - \frac{b}{ay}$

2. $\frac{10a^2y + 3a^3 - 2b^2}{a^2} = 10y + 3a - \frac{2b^2}{a^2}$

3. $\frac{a^2 + ab + b^2}{a} = a + b + \frac{b^2}{a}$

To reduce fractions to a common denominator.

RULE.

Multiply each numerator into all the denominators except its own, for a new numerator of each fraction; and all the denominators into each other, for a new denominator, which will be common to each fraction.

Reduce $\frac{2a}{3}$, $\frac{5a}{b}$, $\frac{4x}{5}$, to fractions having a common denominator.

$$1 \ldots 2a \times b \times 5 = 10ab$$
$$5a \times 3 \times 5 = 75a$$
$$4x \times b \times 3 = 12bx$$

$3 \times b \times 5 = 15b$ for a common denominator. Therefore the new fractions will be,

$$\frac{10ab}{15b} \qquad \frac{75a}{15b} \qquad \frac{12bx}{15b}$$

2...Reduce $\frac{7b^2 - 1}{2b}$, $\frac{4b^2 - b + 2}{2a^2}$.

Ans...$\frac{14a^2b^2 - 2a^2}{4a^2b}$, $\frac{8b^3 - 2b^2 + 4b}{4a^2b}$

To find the greatest common measure of two algebraical quantities.

RULE.

Divide the greater by the less, and divide the preceding divisor by the remainder, repeating the operation until nothing remains; the last divisor will be the common measure required.

1.—Find the greatest common measure of $a^2 + a - 2$ and $a^2 + 2a - 3$.

$$\begin{array}{l}
a^2 + a - 2)\,a^2 + 2a - 3\,(1 \\
\qquad\qquad\quad a^2 + a - 2 \\
\qquad\qquad\quad \overline{\qquad\qquad\quad a - 1)\,a^2 + a - 2\,(a + 2} \\
\qquad\qquad\qquad\qquad\qquad\quad a^2 - a \\
\qquad\qquad\qquad\qquad\qquad\quad \overline{\qquad\quad 2a - 2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 2a - 2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{\;*\quad *\;}
\end{array}$$

The common measure, $a - 1$, is here found by the application of the Rule; but when any factor is contained in every term of either divisor or dividend, which is not found in the other, that factor may be expunged, to facilitate the work, by dividing by it.

2. Find the greatest common measure of $\dfrac{a^4 - b^4}{a^5 - a^3b^2}$

Here we have a^3 in each term of the divisor, but not in the dividend, therefore we cast it out, by dividing the whole divisor by it, which leaves $a^2 - b^2$, and then divide the greater by the less, and again expunge if required.

$$
\begin{array}{l}
a^2 - b^2)a^4 - b^4(a^2 \\
\qquad\quad\;\; a^4 - a^2b^2 \\
\qquad\quad\;\; \overline{} \\
\qquad\quad b^2)a^2b^2 - b^4 \quad \left.\begin{array}{l}\text{Expunge } b^2 \text{ by dividing by} \\ \text{it; our next divisor becomes} \\ a^2 - b^2.\end{array}\right. \\
\qquad\qquad\; a^2 - b^2)a^2 - b^2(1 \\
\qquad\qquad\qquad\qquad\;\; a^2 - b^2 \\
\qquad\qquad\qquad\qquad\;\; \overline{} \\
\qquad\qquad\qquad\qquad\;\;\; * \quad * \\
\qquad\qquad\qquad\qquad\;\; \overline{}
\end{array}
$$

Therefore $a^2 - b^2$ is the greatest common measure sought.

Should the numerical co-efficient of the leading term of the dividend not be divisible by the co-efficient of the divisor, the dividend can be multiplied by such a number as will make it so.

Find the greatest common measure of $\dfrac{3x^2 - 2x - 1}{4x^3 - 2x^2 - 3x + 1}$

Since $3x^2$ cannot measure $4x^3$ exactly, we multiply the denominator by 3, and proceed as before.

$$
\begin{array}{l}
4x^3 - 2x^2 - 3x + 1 \\
3 \\
\hline
\end{array}
$$

$$
\begin{array}{r|l}
3x^2 - 2x - 1) & 12x^3 - 6x^2 - 9x + 3(4x \\
 & 12x^3 - 8x^2 - 4x \\
\hline
 & \qquad 2x^2 - 5x + 3)3x^2 - 2x - 1
\end{array}
$$

Here $2x^2$ does not measure $3x^2$, therefore we multiply by 2.

$$
\begin{array}{l}
3x^2 - 2x - 1 \\
2 \\
\hline
\end{array}
$$

$$
\begin{array}{r|l}
2x^2 - 5x + 3) & 6x^2 - 4x - 2(3 \\
 & 6x^2 - 15x + 9 \\
\hline
 & \qquad 11x - 11
\end{array}
$$

Eliminating 11, by dividing by it, we have a new divisor,

$$
\begin{array}{r|l}
x - 1) & 2x^2 - 5x + 3(2x - 3 \\
 & 2x^2 - 2x \\
\hline
 & \quad - 3x + 3 \\
 & \quad - 3x + 3 \\
\hline
 & \quad * \quad * \\
\hline
\end{array}
$$

Therefore $x - 1$ is the greatest common measure of the two quantities, $3x^2 - 2x - 1$ and $4x^3 - 2x^2 - 3x + 1$. The common measure of three or more quantities can be similarly found, by first finding the measure of two, and then of that and the third, and so on. The common multiple of two quantities can be found, by dividing the product of the proposed quantities by their greatest common measure, and the result will be the least common multiple.

To reduce a fraction to its lowest terms.

RULE.

Divide the numerator and denominator by their greatest common measure, and it is evident that the quotients will form the lowest terms of the given fraction.

1.—Reduce $\frac{2a^3 - 16a - 6}{3a^3 - 24a - 9}$ to its lowest terms.

Ans... $\frac{2}{3}$

2.—Reduce $\frac{6ab^3 + ab^2 - 12ab}{6ab - 8a}$ to its lowest terms.

Ans... $\frac{2b^2 + 3b}{2}$

ADDITION OF FRACTIONS.

RULE.

If the fractions to be added have a common denominator, add their numerators together, as in Arithmetic, placing the common denominator under the numerator; the fraction thus found will be the required one. If the fractions to be added have not all the same denominator, reduce them to such, and proceed as above.

1.—Add together $\frac{a}{d}+\frac{c}{m}$. Having reduced them to

a common denominator, we find $\frac{am}{dm} + \frac{cd}{dm} =$

$\frac{am + cd}{dm}$

2.—Add together $\frac{a + b}{a - b}$ and $\frac{a \quad b}{a + b}$

Ans... $\frac{2a^2 + 2b^2}{a^2 - b^2}$

3.—Add together $\frac{3b + 2}{a}$, $\frac{4b + 3}{d}$, and $\frac{5b + 4}{x}$

Ans... $\frac{d(3b + 2)x + ax(4b + 3) + ad(5b + 4)}{adx}$

4.—Add together $2a - \frac{a + 3}{5}$ and $4a - \frac{2a - 5}{4}$

Ans... $6a - \frac{14a - 13}{20}$

SUBTRACTION OF FRACTIONS.

RULE.

Reduce, as in Addition, and subtract instead of add; the result will be the difference.

1...Find the difference between $\frac{12x}{7}$ and $\frac{3x}{5}$

$$\left.\begin{array}{r} 12x \times 5 = 60x \\ 3x \times 7 = 21x \end{array}\right\} \text{Difference, Ans.} \; \frac{39x}{35}$$

$$7 \times 5 = 35$$

2...Find the difference between $\dfrac{dx}{b - c}$ and $\dfrac{dx}{b + c}$.

Ans...$\dfrac{2dcx}{b^2 - c^2}$

MULTIPLICATION OF FRACTIONS.

RULE.

Multiply the numerators together for a new numerator, and the denominators together for a new denominator; but if the numerator of one, and denominator of the other, can be divided by a quantity common to both, the results can be used, as their values are not changed.

1.—Multiply $\dfrac{4bx}{3}$ by $\dfrac{2b}{5}$ Ans. $\dfrac{4bx}{3} \times \dfrac{2b}{5} = \dfrac{8b^2x}{15}$

2.—Multiply $\dfrac{c + x}{c - x}$ by $\dfrac{x}{c}$ Ans...$\dfrac{x^2 + cx}{c^2 - cx}$

3.—Multiply $\dfrac{a - x^2}{2}$ by $\dfrac{2a}{a - x}$ Ans...$\dfrac{a^2 - ax^2}{a - x}$

DIVISION OF FRACTIONS.

RULE.

Divide the numerator of one by the numerator of the other, and the denominator of one by the denominator of the other, the result is the quotient; but if they are not divisible by each other, invert the divisor, and proceed as in Multiplication.

1.—Divide $\frac{14x^2}{9}$ by $\frac{2x}{3}$

Invert the Divisor, $\frac{3}{2x} \times \frac{14x^2}{9} = \frac{42x^2}{18x} = \frac{7x}{3}$

Or by dividing the numerators and denominators by each other, $\frac{14x^2}{9}$ by $\frac{2x}{3} = \frac{7x}{3}$ for the Answer.

2.—Divide $\frac{4a}{7}$ by $\frac{9a}{5}$

Ans... $\frac{20}{63}$

3.—Divide $\frac{4a+2}{3}$ by $\frac{2a+1}{5a}$

Ans... $\frac{10a}{3}$

4.—Divide $\frac{a^2 - b^2}{a + 2b}$ by $\frac{a - b}{3a + 6b}$

Ans... $3(a + b)$

INVOLUTION.

RULE.

This being only the multiplication of any given quantity continually by itself, to any required power, the operation will be manifest by an Example.

Required the second power of $a + b$, and the third and fourth power of the same.

$$
\begin{array}{l}
a + b \\
a + b \\
\hline
a^2 + ab \\
\qquad ab + b^2 \\
\hline
a^2 + 2ab + b^2 \text{ Second power required.} \\
a + b \\
\hline
a^3 + 2a^2b + ab^2 \\
\qquad a^2b + 2ab^2 + b^3 \\
\hline
a^3 + 3a^2b + 3ab^2 + b^3 \text{ Third power required.} \\
a + b \\
\hline
a^4 + 3a^3b + 3a^2b^2 + ab^3 \\
\qquad a^3b + 3a^2b^2 + 3ab^3 + b^4 \\
\hline
a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \text{ Fourth power.}
\end{array}
$$

The labour of multiplication with binomials, or compound quantities converted into binomials, may in a great measure be avoided by the application of the "Binomial Theorem," discovered by Sir Isaac Newton.

RULE.

A quantity containing two terms is generally called a binomial; in a binomial, the first term is raised to the given power, and the last term is also raised to that power; for example, if $a + b$ be required to be raised to the second power, it will stand thus $(a + b)^2$, which indicates the second power of $a + b$, therefore a will become a^2 for the first term of $(a + b)^2$, and b will become b^2 for the last term; then the intermediate terms are found as follow:—the first term decrease, by unity from the given power, and the second term increase from its first power; therefore, the first being a^2, the second will be ab, and the third will be b^2, the number of terms being always one more than the given index.

Now we have a^2, ab, b^2, to which it is required to prefix numerical co-efficients. The co-efficient of the first term is always one, and if we multiply the co-efficient of a by its index, and divide by the number of terms to that place, we find the co-efficient of the following term. In the case of $(a + b)^2$, the index of the first term is 2, and the co-efficient is 1, therefore $2 \times 1 \div 1 = 2$ for the co-efficient of the second term; now the index of the second term is 1, and the co-efficient is 2, therefore 2×1, and divided by the number of terms to that place, is equal to 1, for the co-efficient of the last. Wherefore, for $(a + b)^2$ we have $a^2 + 2ab + b^2$; all other powers are similarly raised.

Find the 6th power of $a + b$.

Ans...$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.

We may similarly find the co-efficients and indices of the terms of $(a + b)^n$.

Ex. gr....$(a + b)^n$ the first term of which will be a^n; the second will be $na^{n-1}b$; the third will be $\frac{n(n-1)}{2}a^{n-2}b^2$; the fourth will be $\frac{n(n-1)}{2}\ \frac{(n-2)}{3}$ $a^{n-3}b^3$; the fifth will be $\frac{n(n-1)}{2}\ \frac{(n-2)}{3}\ \frac{(n-3)}{4}$ $a^{n-4}b^4$, &c.; and the last term will be b^n. Therefore $(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)}{2}\ \frac{(n-2)}{3}a^{n-3}b^3 + \frac{n(n-1)}{2}\ \frac{(n-2)}{3}\ \frac{(n-3)}{4}a^{n-4}b^4$, &c.......$+ b^n$

EXAMPLES.

1.—Raise $3x - 2y$ to the sixth power.

Ans...$729x^6 - 2916x^5y + 4860x^4y^2 - 4320x^3y^3 + 2160x^2y^4 - 576xy^5 + 64y^6$.

2.—Find the ninth power of $x - y$, or $(x - y)^9$.

Ans...$x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 - 126x^5y^4 + 126x^4y^5 + 84x^3y^6 - 36x^2y^7 + 9xy - y^9$.

3.—Find the third power of $a + b + c$.

Ans... $a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$

This last example being a trinomial quantity, it will be necessary to cast it into a binomial, thus $((a + b) + c)^3$ and proceed precisely as in the others, considering $a + b$ as a single quantity.

It will be requisite to remember that when the last term of a binomial is negative, the alternate terms of the product will be also negative.

Ex. gr....$(a - b)^2 = a^2 - 2ab + b^2$.

Required to raise $(a - b)^4$ to that power.

First we have a^4, a^3b, a^2b^2, ab^3, b^4, by a decreasing, and b increasing, according to the Rule.

Next we have $a^4 - a^3b + a^2b^2 - ab^3 + b^4$, by prefixing the alternate signs.

Next we have 1, $\frac{4 \times 1}{1}$, $\frac{12}{2}$, $\frac{2 \times 6}{3}$, $\frac{1 \times 4}{4}$ for the co-efficients of each, according to the Rule.

Lastly we have $1a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$, by prefixing the co-efficients.

EVOLUTION;

OR, THE

Rule for Extracting the Root of any Quantity.

RULE FOR SIMPLE QUANTITIES.

The root of a quantity is found by dividing its index by the required root; thus, to find the $\sqrt{a^2}$, we divide the index by 2, as is evident, since $\sqrt{a^2} = a^{\frac{2}{2}} = a$, the required root. A surd quantity does not admit of such division: thus, $\sqrt{37}$, $\sqrt[3]{a^2}$, &c., are surds, since the index of either is not divisible by the root. The roots of numerical co-efficients are extracted as in Arithmetic.

Required the $\sqrt[3]{a^3} = a^{\frac{3}{3}} = a.$

Required the $\sqrt[4]{a^4} = a^{\frac{4}{4}} = a.$

Required the $\sqrt{\dfrac{4a^3}{9b^4}} = \dfrac{4a^{\frac{3}{2}}}{9b^{\frac{4}{2}}} = \pm\dfrac{2a^{\frac{3}{2}}}{3b^2}$

Required the $\sqrt[a]{x^b} = x^{\frac{b}{a}}$

Required the $\sqrt[m]{b^{-mn}} = b^{-\frac{mn}{m}} = \dfrac{1}{b^n}$

Required the $\sqrt{5a^4d^{-6}c^{4n}} = 5^{\frac{1}{2}} a^{\frac{4}{2}} d^{-\frac{6}{2}} c^{\frac{4n}{2}} =$
$\pm \sqrt{5}.a^2d^{-3}c^{2n}.$

It is manifest that, by strictly observing the nature of the indices, there is but little difficulty to be met with.

To find the square root of compound quantities.

RULE.

1st.—Range the quantities, as in Division; the square root of the first term will be the first term of the quotient—set its square under the first term, and subtract.

Then bring down two terms to the remainder, for a dividend; set down double the root found for the next divisor; next divide the first term of the new dividend by it, and place the result in both the quotient and the divisor; then multiply the whole divisor by that last term of the quotient, subtracting as usual, and so on until nothing remains, always doubling the quotient for a new divisor.

Find the square root of $a^2 + 2ab + b^2$.

$$
\begin{array}{l|l}
\multicolumn{2}{l}{a^2 + 2ab + b^2(a + b \text{ root required.}} \\
\multicolumn{2}{l}{a^2} \\
\hline
2a + b & 2ab + b^2 \\
 & 2ab + b^2 \\
\hline
 & \quad * \quad\ * \\
\hline
\end{array}
$$

Find the square root of $a^6 + 4a^5 + 2a^4 + 9a^2 - 4a + 4$.

Ans...$a^3 + 2a^2 - a + 2$.

Find the square root of $x^2 + 2xb + b^2 + 2xc + 2bc + c^2$.

Ans...$x + b + c$.

The cube root may be extracted as follows:—Find the cube root of the first term, which root place in the quotient; subtract its cube from the first term of the dividend, and bring down the next three terms; divide the first term of the new dividend by three times the square of the quotient, and place the result in the quotient; then, to three times the square of the first term of the quotient, add three times the first term of the quotient into the second, together with the square of the last term, and it forms the divisor, which multiply by the last term, subtract, and thus proceed until nothing remains.

Find the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

$$
\begin{array}{rl}
 & a^3 + 3a^2b + 3ab^2 + b^3(a + b \\
 & a^3 \\
\hline
3a^2 + 3ab + b^2) & 3a^2b + 3ab^2 + b^3 \\
 & 3a^2b + 3ab^2 + b^3 \\
\hline
 & \quad * \quad\quad * \\
\hline
\end{array}
$$

Otherwise.

Having found the root of the first term, place it in the quotient, and subtract its cube from that term, bringing down the next term for a new dividend. Next involve the root found to its next lower power, multiplying it by the index of the given power for a divisor; divide, and place the result in the quotient; then involve the whole root, subtract and proceed as before.

$$\begin{array}{rl}
 & a^3 + 3a^2y + 3ay^2 + y^3 \; (a + y \text{ required root.} \\
 & a^3 \\
3a^2) & 3a^2y \\
 & a^3 + 3a^2y + 3ay^2 + y^3
\end{array}$$

Find the cube root of $a^6 + 6a^5 + 15a^4 + 20a^3 + 15a^2 + 6a + 1$.

$$(a^2 + 2a + 1$$

$$\begin{array}{rl}
 & a^6 + 6a^5 + 15a^4 + 20a^3 + 15a^2 + 6a + 1 \\
 & a^6 \\
3a^4) & 6a^5 \\
 & a^6 + 6a^5 + 12a^4 + 8a^3 = \text{3rd power of } a^2 + 2a \\
3a^4) & 3a^4 \qquad = \text{remainder.} \\
 & a^6 + 6a^5 + 15a^4 + 2a^3 + 15a^2 + 6a + 1 =
\end{array}$$

3rd power of the quotient.

Any root out of any given quantity can be extracted by this Rule.

EQUATIONS.

It will be requisite here to briefly state the nature of ratio and proportion, as far as regards their utility in connexion with equations.

When two quantities are compared together, that is to say, $a : b$ or $10 : 5$, the first is called the antecedent, and the second the consequent, and the quotient found by dividing the antecedent by the consequent is called the ratio.

If four quantities are proportional, the product of the means is equal to the product of the extremes; in the proportion $a : b :: c : d$, a and d are the extremes, b and c the means. Wherefore, in order to prove the above numerically, they will stand thus,

$10 : 5$ equivalent to $\frac{10}{5} = 2$, the ratio.

Suppose $10 : 5 :: 12 : 6$, the product of the extremes will be 60, and the product of the means will be also 60; therefore the proposition is evident.

To prove that they are in proportion, it will be simply necessary to place them thus, $\frac{10}{5} = \frac{12}{6}$, and, by clearing fractions, we have, $60 = 60$; or the ratio of each $= 2$.

An equation is the equality of two quantities.

The assemblage of quantities at the same side of the sign $=$ is called the member or side; an equation has two sides.

That which is at the left is called the first side, and the other is called the second.

In the equation, $2x + b = a$, the first side is $2x + b$, and the second side is a.

The quantities which compose the same side, when they are separated by $+$ or $-$, are called terms. Thus, the first side of the equation $2x + b = a$ contains two terms, namely, $2x$ and b.

The equation $\frac{3}{4}x + 7 = 8x - 12$ has two terms in each of its sides, namely,

$\frac{3}{4}x$ and $+ 7$ in the first.

$8x$ and $- 12$ in the second.

Although we may have taken the equation $\frac{3}{4}x + 7 = 8x - 12$, by chance, for to serve for example, it ought to be considered (likewise all those of which we shall speak) as coming from a problem of which we can always find an expression, by converting the equation into ordinary language. That of which we have spoken lastly shows how we may find a number such, that in adding 7 to $\frac{3}{4}$ of itself, the sum will be equal to 8 times itself, minus 12.

Similarly, the equation $o \quad bc - cx = ac - bx$, (in which the letters a, b, c, stand to represent known quantities, and x to represent unknown) answers to the following question :—

To find a number (x) such, that in multiplying it by a given number (a), then adding the product of the two given numbers (b and c), and taking away from this sum the product of the given number (c) by the unknown term (x), there may be a result equal to the product of the numbers a and c, minus that of the numbers b and x.

To draw from an equation the value of the unknown quantities, or to have the unknown terms only on one side of the equation, and the known terms on the other, is what we call solving an equation.

The known and unknown quantities can be expressed by any letters we wish, but the unknown are usually expressed by x and y, and the known by the other letters of the alphabet.

Of the solution of equations of the first degree, having only one unknown quantity.

Now, the unknown quantities are found mixed with the known, in three ways :—

1st.—By Addition and Subtraction, as in the Equations,

$$x + 5 = 9 - x$$
$$a + x = b - x$$

2ndly.—By Addition, Subtraction, and Multiplication, as in the Equations,

$$7x - 5 = 12 + 4x$$
$$ax - b = cx - d$$

3rdly.—By Addition, Subtraction, Multiplication, and Division, as in the Equations,

$$\frac{5x}{3} + 8 = \frac{11}{12}x + 9$$

$$\frac{ax}{b} + cx - d = \frac{mx}{n} + \frac{p}{q}$$

In taking the first equation above, for an Example, to solve, (or to find the value of x) it will be necessary to make terms pass from one side of the equation to the other; and, in order to perform this, the terms changing sides must also have their signs changed, the reason of which will appear evident, because changing the signs of a term, when removing it to the other side, is equivalent to subtracting or adding that term to both sides.

1.—Thus $x + 5 = 9 - x$,
becomes $x + x = 9 - 5$, by transposition,
and it becomes $2x = 4$, by Addition and Subtraction,
then $x = 2$, by taking 2 from both sides.

Therefore, a term can be changed from one side of an equation to the other, by also changing its sign, which we call transposition.

In the second we have

2. $a + x = b - x$, which

becomes $a - b = -x - x$, by transposition,

and $a - b = -2x$, by Addition,

then $\dfrac{a - b}{2} = x$, by taking 2 from both sides.

It must be remembered, that taking the co-efficient of the unknown quantity from both sides, is the same as dividing the whole equation by it, therefore, we shall use the word division for the future.

The next example,

3. $7x - 5 = 12 + 4x$,

becomes $7x - 4x = 12 + 5$, by transposition,

and $3x = 17$, by Addition and Subtraction,

then $x = \dfrac{17}{3} = 5\frac{2}{3}$, by dividing by 3.

Again, we have the Example,

4. $ax - b = cx - d$, which

becomes $ax - cx = b - d$, by transposition,

and $(a - c)x = b - d$, $\left\{\begin{array}{l}\text{by placing the co-efficient}\\ \text{of } x \text{ in a parenthesis,}\end{array}\right.$

then $x = \dfrac{b - d}{a - c}$, $\left\{\begin{array}{l}\text{by dividing by the co-efficient}\\ \text{of the unknown quantity.}\end{array}\right.$

The next example contains fractions; and, in order to bring the whole equation to a form that each term shall be a whole number, we must clear it of fractions. This can be performed, either by multiplying the entire equation by each denominator or by finding a common multiple, and then multiplying the whole equation by it, at the same time dividing the product of the numerator of each fraction and the common multiple, by its own denominator.

Here we have,

5. $$\frac{5x}{3} + 8 = \frac{11}{12}x + 9,$$

First it becomes $\frac{5x}{3} - \frac{11}{12}x = 9 - 8$, by transposition,

Next, $5x - \frac{33x}{12} = 27 - 24$, $\left\{\begin{array}{l}\text{by multiplying by first}\\ \text{denominator.}\end{array}\right.$

Next $60x - 33x = 324 - 288$, $\left\{\begin{array}{l}\text{by multiplying by}\\ \text{next denominator,}\end{array}\right.$

Then we have $27x = 36$, by Subtraction,
And $x = \frac{36}{27}$, by dividing by the co-efficient of x.

Lastly we have

6. $$\frac{ax}{b} + cx - d = \frac{mx}{n} + \frac{p}{q}$$

It becomes $\frac{ax}{b} + cx - \frac{mx}{n} = d + \frac{p}{q}$, by transposition,

then $ax + bcx - \frac{bmx}{n} = bd + \frac{bp}{q}$, $\left\{\begin{array}{l}\text{by multiplying}\\ \text{by } b,\end{array}\right.$

next, $anx + bcnx - bmx = bdn + \frac{bnp}{q}$ $\left\{\begin{array}{l}\text{by multiply-}\\ \text{ing by } n,\end{array}\right.$

then,
$anqx + bcnqx - bmqx = bdnq + bnp$, by $\times$ by q,

Then it becomes,
$(anq + bcnq - bmq)x = bdnq + bnp$, by using the (),

lastly, $x = \frac{bdmq + bnp}{anq + bcnq - bmq}$ $\left\{\begin{array}{l}\text{by dividing by the}\\ \text{co-efficient of the}\\ \text{unknown quantity.}\end{array}\right.$

Fractions can be also cleared, by multiplying by a common multiple, and then dividing each fraction by its denominator, as before remarked, thus,

7. $$\frac{2x}{3} + 4 = \frac{4x}{5} + 15 - \frac{5x}{7} \quad \left\{ \begin{array}{l} 105, \text{ common} \\ \text{multiple.} \end{array} \right.$$

It becomes,
$70x + 420 = 84x + 1575 - 75x$, by multiplying by 105,
next, $70x - 84x + 75x = 1575 - 420$, by transposition,
then, $61x = 1155$, by incorporation,
And finally, $x = \frac{1155}{61}$, by dividing by 61.

8. Find the value of y, in the following example,

$$4y - \frac{y-1}{2} = y + \frac{2y-2}{5} + 24,$$

It becomes
$40y - 5y + 5 = 10y + 4y - 4 + 240$ $\left\{ \begin{array}{l} \text{by} \times \text{by a com-} \\ \text{mon multiple,} \end{array} \right.$
$40y - 5y - 10y - 4y = 240 - 4 - 5$, by transposition,
then, $21y = 231$, by Addition and Subtraction,
and $y = 11$, by dividing by the co-efficient of y.

In this example, the sign minus stands before the fraction, therefore the sign or signs must be changed, when the quantity is brought down; the reason is evident, because it means that $4y$ is less by $\frac{y-1}{2}$, and therefore, according to the Rule for Subtraction, the sign must be changed. The same rule is to be observed in all cases, when $-$ stands before a fraction.

9.—What number, when added to $\frac{1}{4}$ of itself, will be equal to 20?

Here we take x for the unknown quantity, therefore we have $x + \frac{1}{3}x = 20$, which becomes,

$3x + x = 60$, by clearing fractions,
then, $4x = 60$, by Addition,
and, $x = 15$, by dividing by the co-efficient of x.

Wherefore, we have 15, added to $\frac{1}{3}$ of itself, equal to 20.

10.—What sum of money, being added to $\frac{2}{9}$ of itself, will be equal to £30 ?

Here we have $x + \frac{2}{9}x = £30$,
it becomes, $9x + 2x = £270$, by clearing fractions,
and $11x = £270$, by Addition,

Therefore, $x = £24$ 10s. $10\frac{10}{11}$d., the Answer.

11.—In the following example, the terms are proportional, therefore the product of the means shall be equal to the product of the extremes.

$$\frac{10 + x}{5} : \frac{4x - 9}{7} :: 14 : 5.$$

We have the product of the extremes. $\frac{50 + 5x}{5} = \frac{56x - 126}{7}$ the product of the means.

then we find, $10 + x = \frac{56x - 126}{7}$, by Division,
next, $70 + 7x = 56x - 126$, by Multiplication,
and, $-56x + 7x = -70 - 126$, by transposition,
and, $-49x = -196$, by Addition and Subtraction,
therefore, $x = 4$, by Division.

Of Questions containing two or more unknown quantities of the first degree.

1.—In the example,

$$12x + 7y = 46,$$
$$8x + 7y = 30,$$

We have $4x = 16$, by Subtraction.
Then, $x = 4$, by Division.

Next, to find the value of y, we can substitute four times the co-efficient of x in either lines, and thus find the value of y.

Taking the first line, we have,

$$48 + 7y = 46,$$

then we have, $7y = 2$, by changing sides and Subt.
Therefore, $y = \frac{2}{7}$, by Division.

It is evident, that, in finding the value of more than one unknown quantity, we must make all the unknown terms disappear, except the one we wish to find first, and then find its value; it will be, therefore, necessary to proceed alternately in finding each value.

There are three or more ways by which the values can be found, the first we have shown; the next will be to find the value of the same unknown quantity, in each equation, and from these equal values find the value of the other.

Therefore in the equation,

2. $$x + y = 6,$$
$$x - y = 2,$$

First, we have, $x = 6 - y$, $\left\{\begin{array}{l}\text{by changing sides and}\\ \text{signs in the 1st equation,}\end{array}\right.$
next, $x = 2 + y$, by similarly changing, in second equation.

Therefore we have $6 - y = 2 + y$ $\left\{\begin{array}{l}\text{by taking each value}\\ \text{of } x \text{ as equals,}\end{array}\right.$
and, $2y = 4$, by changing sides and incorporating.
Then, $y = 2$, by Division.

The valve of x can be similarly found.

3.—We can also find the values, thus,—

$$4x + 2y = 26,$$
$$2x - y = 3,$$

Now, we have, $x + \frac{2y}{4} = \frac{26}{4}$ $\left\{\begin{array}{l}\text{by dividing the top}\\ \text{line by 4;}\end{array}\right.$

then, $x = \frac{26}{4} - \frac{2y}{4}$, by transposition,

and, $\frac{52}{4} - \frac{4y}{4} = y + 3$, $\left\{\begin{array}{l}\text{by substituting the equal}\\ \text{values of } x,\end{array}\right.$

then, $52 - 4y = 4y + 12$, by clearing fractions,

and, $8y = 40$, by transposition and incorporation.

Therefore, $y = 5$, by Division.

4.—A workman labouring for a nobleman, during 12 days, and having his wife and son working with him during the first seven, received 74 shillings; he afterwards worked 8 days, 5 of which he had his wife and son with him, and received for this time, 50 shillings. We demand how much he gained per day for his part, and how much his wife and son together gained in the same time?

Let x be the daily gain of the husband, and
y that of the wife and son;

12 days' work of the husband produces $12x$,
7 of the wife and son equal to $7y$,

Therefore, $12x + 7y = 74$.

Again, 8 days' work of the husband produces $8x$, and 5 days' work of the wife and son equal to $5y$.

Therefore, $8x + 5y = 50$,
and $\quad 12x + 7y = 74$, as above,

We find $60x + 35y = 370$, by $\times$ by co-ef. of y, (5)
and $\quad 56x + 35y = 350$, " " (7)

Then $\quad 4x \quad = 20$, by Subtraction,
and $\quad x = 5$ shillings, the daily gain of the husband.

Again, taking the original equation, namely.

$$\begin{aligned} 8x + 5y &= 50, \\ 12x + 7y &= 74, \end{aligned}$$

We find $96x + 56y = 592$, by $\times$ by co-ef. of x, (8)
and $96x + 60y = 600$, „ „ (12)

Then, $4y = 8$, by subtracting, as before.
And, $y = 2$ shillings, the gain of the wife and son.

This problem could be also solved by finding the value of x or y in each of the equations, and then take each of the answers to form the side of a new equation, from which we could find the value of the remaining quantity, as we have before shown; but the most simple mode appears to be by multiplying each equation by a number, for to make a quantity in both identical.

5.—A courier went from Taunton to Somerton, and back again, in 4 hours, and found that he went at the rate of 16 miles an hour, but returned at the rate of 8 miles an hour. We demand the distance between the two places?

Let x be the time going,
y the time coming back,

It is evident that the rate of returning is only half that of going.

Wherefore $2x = y$, that is, the rate of going is double that of returning.

But the time taken in going and returning is 4 hours, therefore $x + y$ is equal to 4.

Now we have the equation as follows.—

$$\begin{aligned} 2x &= y, \\ x + y &= 4, \end{aligned}$$

Then we have $3x = 4$, by Subtraction,
and $x = 1\frac{1}{3}$, the time taken going to Somerton.

Again, taking the same equations,

$$2x = y,$$
$$x + y = 4,$$

We have, $2x + 2y = 8$, by multiplying the second line by 2, in order to make the first terms identical.

Then we have, $3y = 8$, by subtracting 1st line from 3rd.
and, $y = 2\frac{2}{3}$, the time taken in returning to Taunton.

Now, by multiplying the value of y by the rate o returning, or the value of x by the rate of going, we have the distance required, thus:—

$2\frac{2}{3} \times 8 = 21\frac{1}{3}$, distance between Taunton and Somerton.
$1\frac{1}{3} \times 16 = 21\frac{1}{3}$. " " "

6.—A father, being asked the age of his son, replied: if from double the age he now is, you take three times his age at three years old, you shall have his actual age.

Ans...The child was 9 years old.

7.—A hare is 50 leaps in advance of a grey-hound, and takes 4 leaps to the grey-hound's 3; but 2 of the dog's leaps are equal to 3 of the hare's. We demand how many leaps must the grey-hound take, to catch the hare? Ans...300.

8.—Given, $\frac{1}{2}x + \frac{1}{3}x - \frac{1}{4}x = \frac{1}{2}$, to find x?

Ans...$x = \frac{6}{7}$.

9.—Given, $\sqrt{12 + x} = 2 + \sqrt{x}$, to find x?

Ans...$x = 4$.

10.—Given, $x + y = s$, $x^2 - y^2 = d$, Ans.. $y = \dfrac{s^2 - d}{2s}$, $x = \dfrac{s^2 + d}{2s}$

Equations having three or more unknown quantities may be similarly solved; by first eliminating the same quantity in two of the equations, and then eliminate the same quantity in the next, and either of the others; proceeding similarly as in equations with one or two unknown quantities.

EQUATIONS OF THE SECOND DEGREE.

The equations of which we have treated involve only the first power of the unknown quantity – but if we propose the following question – *To find a number such, that being multiplied by its quintuple, the product shall be equal to* 125—we have a pure equation of the second degree or power. Therefore if we designate this number by x, we find $x \times 5x = 5x^2 = 125$.

Then by dividing by 5, and extracting the square root of both sides of the equation, we have $x = 5$.

We must bear in mind that the square root of Algebraic quantities admits of two signs, *id est*, plus or minus. An example will suffice.

$$\left.\begin{array}{ll}\text{Thus} & (+5)^2 = +25 \\ \text{or} & (-5)^2 = +25\end{array}\right\}\text{according to the rule for Multiplication.}$$

Therefore it is evident that the square root of 25 is either + or − 5.

The next form is that of adfected or compound quantities, that is, involving both the first and second power of the unknown quantities, the remainder being known quantities.

Thus $x^2 + px = q$ is the general formula; x^2 and x being the unknown terms; p and q the known.

Now it is manifest by the 4 Prop. B. 2, that if a line or number be divided into two parts, the sum of their squares shall be less than the square of the sum, by twice the product of the parts—therefore either of the following Rules is required to complete the square of each member of a quadratic equation before its root can be extracted.

RULE I.

Add the square of half the co-efficient of the lowest power of x to both members of the equation, and the squares shall be complete; but if the highest power of x have a co-efficient, reduce it to unity, by dividing the whole equation by it, and then proceed as above.

First $x^2 + px = q$

become $x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} + q$ by adding the square of the half of p to both sides.

RULE II.

Multiply the whole equation by four times the co-efficient of the highest power of x; then add to both members of the product, the square of the original co-efficient of the lowest power of x, and the squares shall be complete. This is called the Hindoo method, and is sometimes more suitable than the first, because the co-efficient of the highest power of x has not to be removed before adding what completes the squares.

1.—Given, $2x^2 + 8x - 20 = 70$, to find x?
It becomes, $16x^2 + 64x - 160 = 560$, by $\times$ by 8,
then, $16x^2 + 64x + 64 = 784$, by + the $(8)^2$ to both, &c.
and, $4x + 8 = 28$. by Evolution.

Ans...$x = 9$.

2.—*To find a number such, that if you multiply it by* 8, *the product shall be equal to the square of the same number, having* 12 *added to it?*

Let x be the number,

$$\begin{aligned} \text{then,}\quad x^2 + 12 &= 8x, \\ x^2 - 8x &= -12, \\ x^2 - 8x + 16 &= 4, \\ x - 4 &= 2, \\ x(= 4 \pm 2) &= 6, \text{ or } = 2. \end{aligned}$$

It is necessary to remember that terms involving the unknown quantity must be made to occupy one side of the equation, before proceeding to find its value; and also, that if we square or extract the root of one side of an equation, we must proceed similarly with the other, in order to keep up the equality between them.

END OF THIRD BOOK.

NUMERICAL PROOF

OF

EUCLID'S SECOND BOOK.

Let numbers be placed on each given line, according to its divisions, named in the enunciation.

Prop. 1.... $\frac{20}{10 \cdot 10 \cdot 10}$. Now, proceeding to prove the Proposition as directed by the enunciation, we have,

$$20 \times 30 = 600.$$

Again, $10 \times 20 + 10 \times 20 + 10 \times 20 = 600$.

Therefore, each side being equal to 600, it is evident that the rectangle contained by the two right lines (supposed to contain 20 and 30 units respectively) is equal to the rectangles contained by the undivided line (20) and the several parts of the divided line. We shall similarly prove the other Propositions.

Prop. 2.... $\underline{10 \cdot \quad 20}$

$(30)^2 = 900$. And $30 \times 10 + 30 \times 20 = 900$.

Prop 3.... $\underline{10 \cdot \quad 20}$

$30 \times 10 = 300$. And $(10)^2 + 20 \times 10 = 300$.

Prop. 4.... $\underline{10 \cdot \quad 20}$

$$(10)^2 + (20)^2 + 2 \times 10 \times 20 = 900.$$

And $(30)^2 = 900$.

Prop. 5.... $\frac{15 \cdot \quad 15}{10 \cdot \quad 20}$

$10 \times 20 + (5)^2 = 225$. Again, $(15)^2 = 225$.

Prop. 6.... 10· 10· 10
$30 \times 10 + (10)^2 = 400$. And $(20)^2 = 400$.

Prop. 7.... 20· 10
$(30)^2 + (20)^2 = 1300$. But $2 \times 30 \times 20 + (10)^2 = 1300$

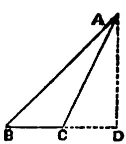
Prop. 8.... 20· 10
$4 \times 30 \times 10 + (20)^2 = 1600$. And $(40)^2 = 1600$.

Prop. 9.... 15· 15 / 10· 20
$(10)^2 + (20)^2 = 500$. And $2(15)^2 + 2(5)^2 = 500$.

Prop. 10.... 10· / ·10·10·
$(30)^2 + (10)^2 = 1000$. And $2(10)^2 + 2(20)^2 = 1000$.

Prop. 11....This Problem cannot be numerically proved.

Prop. 12....Suppose BC $= 10$, CD $= 10$, and DA $= 20$, then,



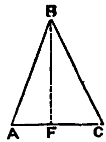
$$\begin{array}{rlcrl} (BA)^2 &= 800 & & (BC)^2 &= 100 \\ -2BC.CD &= 200. & \text{And} & (CA)^2 &= 500 \\ \hline \text{Excess,} & 600 & & \text{Sum,} & 600 \end{array}$$

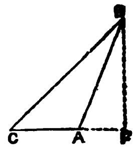
Prop. 13....Assume AC $= 30$, AF $= 20$, and BF $= 50$, then,

$$\begin{array}{rlcrl} (AB)^2 &= 2900 & & (BC)^2 &= 2600 \\ (AC)^2 &= 900. & \text{And} & 2 \times 20 \times 30 &= 1200 \\ \hline \text{Sum of squares,} & 3800 & & \text{Sum,} & 3800 \end{array}$$

2nd Case.

The perpendicular falling outside.

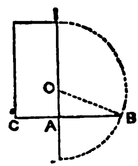


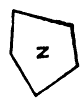


$$(CB)^2 = 5000 \qquad (AB)^2 = 2900$$
$$(CA)^2 = 900. \qquad \text{And } 2.30.50 = 3000$$

Sum of squares, 5900   Sum, 5900

Prop. 14....Let OI be equal to 10, and OA = 4, then CA or AL will be = 6.





Therefore, according to the Problem, we have $14 \times 6 + (4)^2 = 100 = (OB)^2$; then, by taking away the common square OA = 16, we have 84, equal to the square AB, and also equal to the rectangle under IA, AC, or 14×6, which is equal to the given figure Z.

ALGEBRAIC PROOF

OF

EUCLID'S SECOND BOOK.

We shall denote the lines, and their divisions, by the small letters of the alphabet, and place the same letter on equals.

Prop. 1.... $\overline{a \mid b \mid c}^{\,d}$. Now, it is manifest, that the divided line must be denoted by $a + b + c$; therefore, taking these divisions as a single quantity, and multiplying it by d, produces the rectangle under the two lines; but if d be multiplied into the several parts, the sum of the results shall be equal to that rectang'e.

Therefore, we have,

$$(a + b + c)d = ad + bd + cd,$$

Then, $ad + bd + cd = ad + bd + cd$, by $\times$.

Prop. 2....Dividing the line thus, $\underline{a \mid b}$, we have,

$$(a + b)^2 = a(a + b) + b(a + b),$$

then, $a^2 + b^2 + 2ab = a^2 + b^2 + 2ab$, by Add. and Invo.

Prop. 3....Let the given line be, $\underline{a \mid b}$, and, according to the enunciation, we have,

$$(a + b)b = b^2 + ab,$$

Hence it becomes, $ab + b^2 = b^2 + ab$, by $\times$.

Prop. 4.... $\underline{a \mid b}$, $a^2 + b^2 + 2ab = (a + b)^2$.
Then, $a^2 + b^2 + 2ab = a^2 + b^2 + 2ab$, by Involution.

Prop. 5....Here we must divide the line into two equal and two unequal parts. $\frac{a \mid a}{d \mid}$

$$(2a - d)d + (a - d)^2 = a^2,$$

then, $2ad - d^2 + a^2 - 2ad + d^2 = a^2$, by $\times$ and Invo.
And $a^2 = a^2$, by incorporation.

Prop. 6.... $\underline{a \mid a \mid b}$. We have as follows,
$(2a + b)b + a^2 = (a + b)^2$,
$a^2 + 2ab + b^2 = a^2 + 2ab + b^2$, by + and Invo.

Prop. 7....By dividing the line into two parts, $\underline{a \mid b}$
We have $(a + b)^2 + b^2 = 2(a + b)b + a^2$,
then it becomes, $a^2 + 2ab + 2b^2 = a^2 + 2ab + 2b^2$,
by Involution and Multiplication.

Prop. 8....We have the line $\underline{a \mid b}$, therefore,
$(a + 2b)^2 = 4(a + b)b + a^2$,
and $a^2 + 4ab + 4b^2 = a^2 + 4ab + 4b^2$, by Involution and Multiplication.

Prop 9.... $\underline{a \mid b \mid a \mid b}$.
$a^2 + (b + a + b)^2 = 2(a + b)^2 + 2(b)^2$,
then we have, $2a^2 + 4b^2 + 4ab = 2a^2 + 4b^2 + 4ab$,
by Involution and Multiplication.

Prop. 10.... $\underline{a \mid a \mid d}$.
$(a + a + d)^2 + d^2 = 2a^2 + 2(a + d)^2$,
then, $4a^2 + 4ad + 2d^2 = 4a^2 + 4ad + 2d^2$, by Involution and Multiplication.

Prop. 11, requires to be solved by decimals, therefore the parts cannot be expressed by any rational number.

Prop. 12....Suppose BC $= a$, CD $= b$, and DA $= c$,
then we have $(a + b)^2 + c^2 = (\text{AB})^2$, by Prop. 47, .B 1,
again, ,, $b^2 + c^2 = (\text{CA})^2$, ,, ,,
therefore, $a^2 + 2ab + b^2 + c^2 = a^2 + 2ab + b^2 + c^2$.

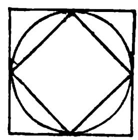
The 13th Prop. can be similarly proved. And if, by reference to the Numerical Proof of the 14th, we substitute letters for the numbers assumed, its solution will be also evident.

FOURTH BOOK.

DEFINITIONS.

1. A rectilineal figure is said to be inscribed in a circle, when each of its angles touches the circumference of the circle.

2. A rectilineal figure is said to be circumscribed about a circle, when each of its sides touches the circle.



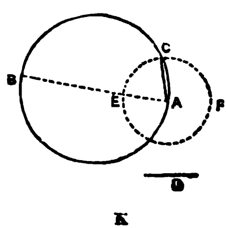
3. A circle is said to be inscribed in a rectilineal figure, when each of the sides of the figure touches its circumference.

4. A circle is said to be circumscribed about a rectilineal figure, when each of the angles of the figure touches its circumference.

5. A right line is said to be inscribed in a circle, when its extremities are in the circumference of the circle.

PROPOSITION I. PROBLEM.

In a given circle (BCA) to inscribe a given right line (D) which is not greater than the diameter of the circle.

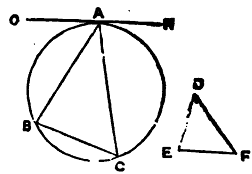


Draw a diameter AB of the given circle, and if this be equal to the given line D, that which is proposed is done. If not, assume in it AE, equal to D (by Prop. 3, B. 1), and from the centre A, with the interval AE, describe a circle ECF, and to either of its

intersections, with the given circle, draw the right line AC, this will be equal to AE (by Def. 15, B. 1), and therefore equal to the given line D (by Const. and Ax. 1).

PROPOSITION II. PROBLEM.

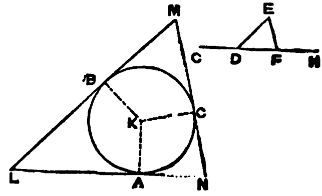
In a given circle (BAC) to inscribe a triangle, equianglar to a given triangle (EDF).



Draw the right line GH touching the given circle in any point A; and at the point A, with the line AH, construct the angle HAC, equal to E; and at the same point, with the right line AG, construct GAB equal to F, and draw BC. Because the angle E is equal to the angle HAC (by Const.), and the Angle HAC is equal to the angle B in the alternate segment (by Prop. 32, B. 3), the angles E and B are equal; similarly, F and C are equal; therefore the remaining angle D is equal to the remaining angle BAC (by Cor. 2, Prop. 32, B. 1); and therefore the triangle BAC, inscribed in the given circle, is equiangular to the given triangle EDF.

PROPOSITION III. PROBLEM.

About a given circle (ABC) to circumscribe a triangle, equiangular to a given triangle (DEF).

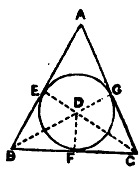


Produce any side DF of the given triangle both ways, to G and H; find the centre K of the given circle, and draw the right line KA; make, at the point K, and with the right line

KA, an angle BKA equal to the angle EDG: and at the same point K, make, at the other side of KA, an angle AKC equal to the angle EFH, and draw the right lines LM, LN, and MN, touching the circle in the points B, A, and C. Therefore, since the four angles of the quadrilateral figure LBKA are equal to four right angles (by Cor. 6, Prop. 32, B. 1), and the angles KBL and KAL are right angles (by Prop. 18, B. 3), the remaining angles AKB and ALB will be together equal to two right angles, but the angles EDG and EDF together are equal to two right angles (by Prop. 13, B. 1), therefore the angles AKB and ALB together are equal to the angles EDG and EDF taken together; but AKB and EDG are equal (by Const.), and therefore ALB and EDF are equal. Similarly it can be demonstrated that the angles ANC and EFD are equal; therefore the remaining angle M is equal to the remaining angle E (by Cor. 2, Prop. 32, B. 1). therefore the triangle LMN, circumscribed about the given circle ABC, is equiangular to the given triangle.

PROPOSITION IV. PROBLEM.

To inscribe a circle in a given triangle (BAC).

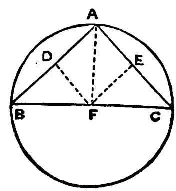


Bisect any two angles B and C, by the right lines BD and CD, and from the point in which these right lines meet, draw to any of the sides of the triangle BAC a perpendicular DF; the circle described from the centre D, and with the interval DF, will be inscribed in the given triangle. For draw DE and DG perpendicular to BA and AC; and since in the triangles DEB, DFB, the angles DEB and DBE are equal to the angles DFB and DBF (by Const.), and the side DB is common to both, therefore the sides DE and DF will be equal (by Prop '6 B 1); it can be similarly

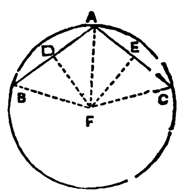
demonstrated that DG and DF are equal; therefore the three lines DE, DF, and DG, are equal in themselves (by Ax. 1), and therefore the circle described from the centre D and with the interval DF passes through E and G, and since the angles at F, E, and G are right, the right lines BC, BA, and AC touch the circle (by Prop. 16, B. 3), and therefore the circle FEG is inscribed in the given triangle.

PROPOSITION V. PROBLEM.

To circumscribe a circle about a given triangle (BAC).



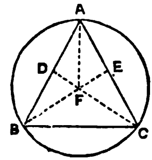
Bisect any two sides BA and AC of the given triangle, in D and E; and through D and E draw DF and EF perpendicular to AB and AC; and from their point of meeting F draw to any angle A of the triangle BAC, the right line FA; the circle described from the centre F, with the interval FA, shall be circumscribed about the given triangle.



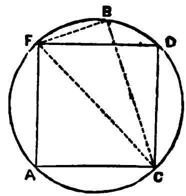
For draw FB and FC; and since in the triangles FDA and FDB, the sides DA and DB are equal (by Const.), but FD is common to both, and the angles at D right (by Const.), the sides FA and FB shall be equal (by Prop. 4, B. 1). It can be similarly demonstrated that the right lines FA and FC are equal, therefore the three right lines, FA, FB, and FC are equal (by Ax. 1), and therefore the circle described from F, with the interval FA, shall pass through B and C, and therefore is circumscribed about the given triangle BAC.

SCHOL.—This Problem is the same as to descri circle through three given points not placed *in directu*

Cor. 1.—If the centre F fall within the triangle, it is evident that all the angles are acute, for each is in a segment greater than a semi-circle (by Prop. 31, B. 3).* But if the centre F be in any side of the triangle,† the angle opposite to that side will be right, for it is an angle in a semicircle (by Prop. 31, B. 3). And if the centre fall without the triangle,‡ the angle opposite to that side towards which the centre falls, will be obtuse, because it is an angle in a segment less than a semicircle (by Prop. 31, B. 3).



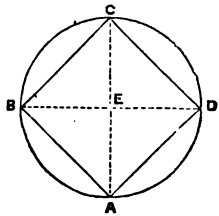
Cor. 2.—A circle can be described about a quadrilateral figure AFDC, whose opposite angles are equal to two right angles. For draw the diagonal, and the circle described about the triangle FAC shall pass through D. For take, in the arch FDC, any point B, and draw BF and BC; and in the quadrilateral figure AFBC inscribed in a circle, the angles A and B will be equal to two right angles (by Prop. 22, B. 3); but D and A are equal to two right angles (by Hypoth.), therefore B and D are equal, and since either of them B is in the circumference, the remaining angle D will be in the same circumference (by Cor. Prop. 21, B. 3), and therefore the circle passes through D, and is circumscribed about the quadrilateral figure AFDC.



* See adjoining Figure. † See First Figure.
‡ See Second Figure.

PROPOSITION VI. PROBLEM.

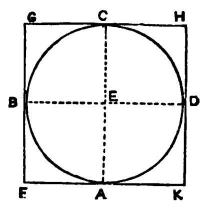
To inscribe a square in a given circle (ABCD).



Draw the diameters AC, BD, to the given circle, at right angles to each other, and drawing AB, BC, CD, DA, the square ABCD will be inscribed in the given circle. For since the angles at the centre E are right, and therefore equal, the arches on which they stand will be equal (by Schol. Prop. 29, B. 3), and therefore their subtending lines are equal; ABCD is therefore equilateral; and because BD is the diameter of the given circle, BAD will be an angle in a semicircle, and therefore right (by Prop. 31, B. 3). It can be similarly demonstrated that the remaining angles B, C, and D are right angles; and since the sides are also equal, ABCD shall be a square.

PROPOSITION VII. PROBLEM.

To circumscribe a square about a given circle (ABCD).



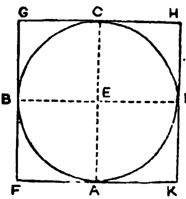
Draw two diameters, AC and BD, of the given circle, at right angles to each other, and through their extremities, A, B, C and D, draw the right lines KF, FG, GH, HK, touching the circle, the figure FGHK will be a square circumscribed about the circle. For since EA is drawn from the centre to the point of contact, the angle EAF will be a right angle (by Prop. 18, B. 3), but the angle AEB is right (by Const.), and therefore the right lines FK and BD are parallel (by Prop. 28, B. 1); it can be similarly shown that GH is parallel to BD, and also that FG and KH are parallel to AC, therefore GD, BK, FC, and AH are parallelograms; and because the angles at A are right angles.

the angles at G and H, opposite to them, will be right (by Prop. 34, B. 1). And it can be similarly demonstrated that K and F are right angles, therefore FGHK a rectangle; and since AC and BD are equal, and FK and GH are each equal to BD, and FG and KH are each equal to AC (by Prop. 34, B. 1), it appears that FGHK is also equilateral, and therefore a square.

Cor.—A square circumscribed is double the square inscribed in the same circle; because it is equal to the square of the diameter of the circle; but it appears, from Proposition 47, Book 1, that the square of the diameter is double of the inscribed square.

PROPOSITION VIII. PROBLEM.

To inscribe a circle in a given square (FGHK).

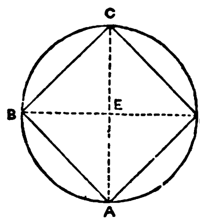


Bisect two adjacent sides, GH and FG, of the given square, in C and B, through C draw CA parallel to either FG or KH, and through B draw BD parallel to either GH or FK; the circle described from the centre E, with the interval EC, will be inscribed in the given square. For since GE, EH, and EF are parallelograms (by Const.), their opposite sides will be equal (by Prop. 34, B. 1); hence, since the equals GH and FG are bisected (by Const.), their halves GC and GB are equal; CE and EB will be equal to GB and GC (by Prop. 34, B. 1), and also equal in themselves; but the remaining sides ED and EA are equal to CE and EB, because they are equal to CH and BF, which are the halves of GH and FG (by Prop. 34, B. 1), therefore the squares EC, EB, EA, and ED are equal to one another, and therefore the circle described from the centre E, with the interval EC, shall pass through B, A, and D; and since the angles at C, B, A and D are right angles, the circle itself touches the sides of the

given square (by Prop. 16, B. 1), and therefore is inscribed in it.

PROPOSITION IX. PROBLEM.

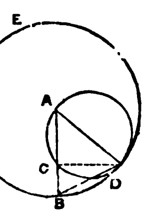
To circumscribe a circle about a given square (ABCD).



Draw AC and BD, intersecting each other in E; the circle described from the centre E, through A, shall pass through B, C, and D. For because in the isosceles triangle ABC the angle B is right, the remaining angles will be each half a right angle (by Cor. 3, Prop. 32, B. 1); it can be similarly demonstrated that all the angles into which the angles of the square are divided by AC and BD are each half a right angle, and therefore are equal to each other; therefore in the triangle AEB, since the angles EAB and EBA are equal, the sides EA and EB are equal (by Prop. 6, B. 1). In the same manner it can be demonstrated that the remaining sides ED and EC are equal to EA and EB, therefore the four lines EA, EB, EC, and ED are equal; therefore the circle described from the centre E, through A, shall pass through B, C and D, and therefore will be circumscribed about the given square.

PROPOSITION X. PROBLEM.

To construct an isosceles triangle, having each angle at the base, double of the vertical angle.



Take any right line AB, and divide it in C, so that the rectangle under AB and CB be equal to the square of AC (by Prop. 11, B. 2), and from the centre A, with the interval AB, describe the circle BED, in which inscribe the right line BD equal to AC (by Prop. 1, B. 4); draw AD, BAD will be an isosceles triangle,

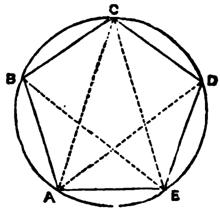
and each angle B and D will be double the angle A. For draw DC, and circumscribe the circle ACD, about the triangle DCA. Since the rectangle under AB and BC is equal to the square of AC (by Const.), and therefore to the square of BD, which is equal to AC; the right line BD touches the circle ACD (by Prop. 37, B. 3), and therefore the angle BDC is equal to A in the alternate segment (by Prop. 32, B. 3), add to both CDA, and BDA will be equal to the sum of CDA and A; but because the sides AB and AD are equal the angles B and BDA will be equal, therefore the angle B is equal to CDA and A taken together; but the external angle BCD is equal to the angles CDA and A (by Prop. 32, B. 1), therefore the angles B and BCD are equal, and therefore the sides BD and CD are equal (by Prop. 6, B. 1), but BD and CA are equal (by Const.), therefore CD and CA are equal, and therefore the angles A and CDA are equal; but the angle BDA is equal to A and CDA taken together, therefore it is double of A, and therefore the angle B is likewise double of A.

SCHOL.—It appears that the triangle BDC is isosceles, of which each angle at the base BC is double the vertical angle; and that the triangle ACD is isosceles, of which the angle at the vertex ACD is triple of each angle at the base, for it is equal to B and BDC taken together (by Prop. 32, B. 1), of which one B is double of the angle A, and the other BDC equal to it.

COR.—Hence appears a method, upon a given right line BD, for a base, to construct an isosceles triangle, having the angles at the base double the vertical angle; for construct such a triangle CDB, in which the given line BD may be a side; and produce BC, make the angle BDA equal to the angle B, BA and DA meet, and that required shall be done.

PROPOSITION XI. PROBLEM.

To inscribe an equiangular and equilateral pentagon in a given circle (ABCDE).



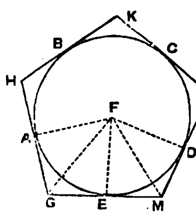
Construct an isosceles triangle having each angle at the base double of the vertical angle (by Prop. 10, B. 4) and in the given circle inscribe ACE equiangular to it (by Prop. 2, B. 4), bisect the angles at the base, A and E, by the right lines AD and EB, and draw AB, BC, CD, DE, and EA. Therefore because each of the angles CAE and CEA is double of ECA (by Const.), and are bisected by the right lines AD and EB, the five angles CEB, BEA, ACE, CAD, and DAE shall be equal to one another, and therefore the arches upon which they stand are equal (by Schol. Prop. 29, B. 3), and therefore the right lines CB, BA, AE, ED, and DC, subtending these arches, are also equal (by same Schol.), and therefore the pentagon ABCDE is equilateral. But since the arches AB and DE are equal, by adding the common arch BCD to both, the arch ABCD will be equal to the arch BCDE, therefore the angles AED and BAE standing upon them are also equal (by Schol. Prop. 29, B. 3), and it can be similarly demonstrated that all the remaining angles are equal, and therefore the pentagon ABCDE is also equiangular.

Cor. 1. Hence it appears that all equilateral figures inscribed in a circle, are also equiangular.

Cor. 2.—An equilateral and equiangular pentagon can be constructed upon a given line AE, by constructing upon it as a base, an isosceles triangle ACE, in which each of the angles at the base is double of the vertical angle (by Cor. Prop. 10, B. 4), and about it circumscribing a circle, the pentagon inscribed in this shall be constructed upon the given line.

PROPOSITION XII. PROBLEM

To circumscribe an equiangular and equilateral pentagon about a given circle (ABCDE).



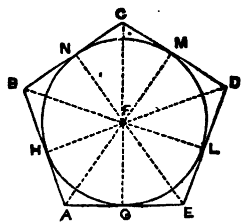
Let the points A, B, C, D, and E be the vertices of the angles of an equilateral and equiangular pentagon inscribed in the given circle, and draw GH, HK, KL, LM, and MG, touching the circle in these points, and GHKLM shall be an equilateral and equiangular pentagon circumscribed about the given circle. For draw FA, FG, FE, FM, and FD; and since in the triangles FGA, FGE, the sides GA and GE are equal, for they are touching the circle from the same point G, and the sides FA and FE are also equal, but F G is common to both, the angles FGA and FGE will be equal, and also AFG and EFG (by Schol. Prop. 8, B. 1), therefore AGE is double of FGE, and AFE double of GFE; it can be similarly demonstrated that DME is double of FME, and also DFE of MFE; but since the arches AE and ED are equal (by Const.) the angles AFE and EFD are equal (by Schol. Prop. 29, B. 3), and therefore the halves of them GFE and MFE are equal; but the angles FEG and FEM are also equal, and the side EF is common, therefore FGE and FME are equal, and the sides GE and EM (by Prop. 26, B. 1); therefore the right line GM is double of GE; it can be similarly demonstrated that GH is double of GA, but GE and GA are equal, and therefore GM and GH are also equal; it can be similarly demonstrated that the remaining sides are equal, therefore the pentagon GHKLM is equilateral: but since the angles DME

and AGE are double of FME and FGE, and FME is equal to FGE, DME will be also equal to AGE; and it can be similarly demonstrated that the remaining angles are equal, and therefore GHKLM is also equiangular.

SCHOL.—Any equilateral and equiangular figure can be similarly circumscribed about a given circle, by first inscribing such a figure, and then drawing tangents through its angles.

PROPOSITION XIII. PROBLEM.

To inscribe a circle in a given equilateral and equiangular pentagon (ABCDE).



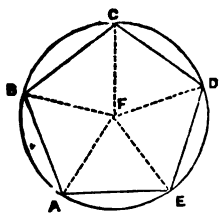
Bisect any two adjacent angles A and E by two right lines AF and EF meeting in F, and from F draw FG perpendicular to AE; the circle described from the centre F, with the interval FG, will be inscribed in the given pentagon. For draw FB, FC, and FD, and from F, to the sides, let fall the perpendiculars FH, FN, FM, and FL. Therefore since in the triangles AFB, AFE, the sides AB and AE are equal (by Hypoth.), and AF common, and the angles FAB and FAE equal (by Const.), the angles ABF and AEF shall be also equal (by Prop. 4, B. 1), but ABC and AED are equal (by Hypoth.), and therefore because AEF is half of AED (by Const.), ABF is also half of ABC; it can be similarly demonstrated that the remaining angles are bisected by right lines drawn from F; therefore in the triangles FBH and FBN, the angles FBH and FBN are equal, but the angles H and N are right angles, and the side FB, to which H and N are opposite, is

common to both, therefore the sides FH and FN are equal (by Prop. 26, B. 1); and it can, in the same manner, be demonstrated that all the perpendiculars are equal, therefore the circle described from the centre F, through G, shall pass through H, N, M, and L, and on account of the angles at G, H, N, M, and L being right angles, the sides of the given pentagon touch the circle at those points.

SCHOL.—Hence by the same method a circle can be incribed in any equilateral and equiangular figure.

PROPOSITION XIV. PROBLEM.

To describe a circle about a given equiangular and equilateral pentagon (ABCDE).



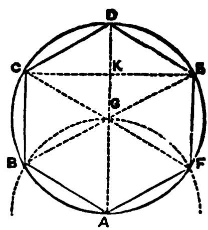
Bisect the angles A and E by the right lines AF and EF, and from the meeting point F, through either points A or E describe a circle; this shall also pass through B, C, and D. For draw FB, FC, and FD, and in the triangles FAE and FAB, since the sides FA, AE, are equal to the sides FA, AB, and the angle FAE is equal to the angle FAB (by Const.), the angles FBA and FEA shall be also equal (by Prop. 4, B. 1); therefore since the angles ABC and AED are equal (by Hypoth.), and FEA is half of AED (by Const.), FBA will be also half of ABC, and therefore ABC is bisected by FB; it can be similarly demonstrated that the remaining angles C and D are bisected; therefore because in the triangle AFE, the angles FAE and FEA are equal, for they are the halves of the equal angles BAE and AED, and therefore the sides FE and FA are equal (by Prop. 6, B 1), and it can be similarly demonstrated

that all the remaining right lines FB, FC, and FD are equal, therefore the five lines FA, FB, FC, FD, and FE are equal; and therefore the circle described from the centre F, through A, shall pass through B, C, D, and E, and is circumscribed about the given pentagon.

Schol.—Hence by the same method a circle can be circumscribed about any equilateral and equiangular figure.

PROPOSITION XV. PROBLEM.

To inscribe an equilateral and equiangular hexagon in a given circle (ABCDEF).



Assume the centre G of the given circle, and draw the diameter AGD, from the centre A, describe a circle through G, and through the intersections B and F draw BE and FC diameters to the given circle: drawing AB, BC, CD, DE, EF, and FA, the equilateral and equiangular hexagon ABCDEF shall be inscribed in the given circle. For because AB and AG are the radii of the same circle BGF, they are equal, and since GA and GB are the radii of the same circle ABCDEF, they are equal; the triangle BGA is therefore equilateral, and therefore the angle BGA is one third of two right angles (by Cor. 4, Prop. 32, B. 1); AGF is similarly equilateral, and the angle AGF is also a third part of two right angles; but BGA and AGF together with FGE are equal to two right angles (by Cor. 1, Prop 13, B. 1), and therefore FGE is one third part of two right angles; therefore the three angles BGA, AGF, and FGE are equal, and therefore the vertical angles EGD, DGC, and CGB are equal, and therefore the six angles at G are equal, and therefore the arches on which they stand are also equa.

and therefore the right lines subtending these arches are also equal, (by Schol. Prop. 29, B. 3), therefore the hexagon is equilateral; and therefore, since it is inscribed in a circle, it is also equiangular (by Schol. Prop. 11, B. 4).

Cor. 1.—The side of a hexagon is equal to the radius of the circle in which it is inscribed.

Cor. 2.—By bisecting the arches AB, BC, a figure of twelve sides might be inscribed, and again bisecting these, a figure of twenty-four sides, and so on.

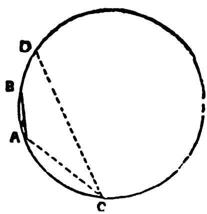
Cor. 3.—An equilateral and equiangular hexagon can be described upon a given line BA, by describing an equilateral triangle BGA upon it, and from the centre G describing a circle through A, and inscribing a hexagon in it.

Cor. 4.—Drawing AC, AE, and CE, an equilateral triangle will be inscribed in the given circle, whose sides bisect the radii perpendicular to them. For let GD be perpendicular to any side CE, and in the triangles DCK, GCK, the angles at K are right, and the angles DCK and GCK standing on equal arches DE and EF are equal (by Schol. Prop. 29, B 3), and the side CK between the equal angles common, therefore the sides DK and GK are equal (by Prop. 26, B. 1), and therefore GD is bisected.

Cor. 5.—Hence it follows that the square of the side of an equilateral triangle is triple the square of the radius of the circle in which it is inscribed; for if AC be drawn, its square will be equal to the squares of AG and GC together with twice the rectangle under AG and GK (by Prop. 12, B. 2); but since GK is half of DG (by Cor. pr.), twice the rectangle under AG and GK is equal to the square of AG, therefore the square of AC is equal to twice the square of AG, together with the square of GC, that is, to triple the square of the radius.

PROPOSITION XVI. PROBLEM.

To inscribe an equilateral and equiangular quindecagon in a given circle (CAD).



Let CD be the side of an equilateral triangle inscribed in the circle CAD, and also let CA be the side of an equilateral and equiangular pentagon inscribed in the circle CAD, bisect AD, and AB will be the side of the required quindecagon. For if the whole circumference be divided into fifteen parts, because the arch CD is one third part of the whole circumference, it contains five of these parts in it, and similarly the arch CA contains three of them, and therefore AD contains two, therefore the arch AB is the fifteenth part of the whole circumference, and therefore AB is the side of the quindecagon.

END OF FOURTH BOOK.

FIFTH BOOK.

DEFINITIONS.

1. A less magnitude is called an aliquot part, or a submultiple of a greater, when the less measures the greater.

2. A greater is called a multiple of a less, when the greater is measured by the less.

3. Ratio is a mutual relation of two magnitudes of the same kind, in respect of quantity.

4. Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the greater.

5. Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when any submultiple whatsoever of the first is contained in the second, as often as an equi-submultiple of the third is contained in the fourth.

6. Magnitudes which have the same ratio are called proportionals.

7. When a submultiple of the first is contained oftener in the second, than an equi-submultiple of the third is contained in the fourth, then the first is said to have a less ratio to the second, than the third has to the fourth; and conversely, the third is said to have a greater ratio to the fourth, than the first to the second.

8. Proportion is the similitude of ratios.

9. And proportion consists of three terms, at least.

10 When three magnitudes are proportional (A to B as B to C), the first is said to have to the third (A to C) a duplicate ratio of that which it has to the second, namely, of the ratio of A to B.

11. When four magnitudes are proportional, one after the other (A to B as B to C, and B to C as C to D), the first is said to have to the fourth (A to D) a triplicate ratio of that which it has to the second (namely, of the ratio of A to B). And so on, increasing by unity, whatsoever be the number of proportionals.

12. If there be any number of magnitudes of the same kind (A, D, C, and F), the first is said to have to the last (A to F) a ratio compounded of the ratios which the first has to the second, and the second to the third, and the third to the fourth (A to D, D to C, and C to F), and so on to the last.

13. In proportionals, the antecedents are called homologous to the antecedents, and the consequents to the consequents.

Either the order or the magnitude of proportionals can be changed, so that they remain still proportionals; the various modes of changing, which are made use of by Geometers, are designated by the following names:—

14. By Permutation, when it is inferred that if there be four magnitudes, of the same kind, proportional, the first is to the third as the second to the fourth; *as is shown in Prop.* 33, *B.* 5. Ex. gr.—A to B as C to Do, becomes A to C as B to Do.

15. By Inversion, when it is inferred that if four magnitudes be proportional, the second is to the first as the fourth to the third; *as is demonstrated in Prop.* 20, namely—A to B as CD to E, becomes B to A as E to CD.

16. By composition, when it is inferred, if four magnitudes be proportional, that the first together with the second is to the second, as the third together with

the fourth is to the fourth; *as shown in Cor. Prop.* 21. Thus—A to B as C to D, will be A + B to B as C + D to D.

17. By Division, when it is inferred, if there be four magnitudes proportional, that the difference between the first and the second is to the second as the difference between the third and the fourth to the fourth; *as is proved in Cor.* 2, *Prop.* 25. Ex. gr.—A to B as C to D, becomes A — B to B as C — D to D.

18. By Conversion, when it is inferred, that if there be four magnitudes proportional, the first is to the sum or difference of the first and second, as the third is to the sum or difference of the third and fourth. See Props. 21 and 25, also Cor. 1, Prop. 25. Thus—A is to B as C is to D, becomes A to (A + B) as C to (C + D), or A to (A — B) as C to (C — D).

19. Ex æquali vel ex æquo (or from equality), when it is inferred, that if there be many magnitudes, and an equal number of others, which, if taken two by two, are in the same ratio, the first is to the last in the first series, as the first to the last in the next series.

Of this there are the two following species:—

20. Ex æquo ordinate (from orderly equality), when the first magnitude will be to the second in the first series, as the first to the second in the next series, and the second to the third in the first series, as the second to the third in the next, and so on; and it is inferred, as in Def. 19, that the first is to the last in the first series, as the first to the last in the next series; *as shown in Prop.* 34. Ex. gr.—Let A, B, and CO, be three magnitudes in the first series, and D, E, and F, be three in the second series; then, according to the Definition, we have

A to B as D to E,
and B to CO as E to F,
and A to CO as D to F,

And so on, if there were four or more magnitudes.

21. Ex æquo perturbate (or disorderly proportion), when the first magnitude will be to the second in the first series, as the last but one to the last in the next; and the second to the third in the first series, as the last but two to the last but one in the next, and so on; and as above, that the first is to the last in the first series, as the first to the last in the next series; *as shown in Prop.* 38. Thus,

Let there be A, B, and CO,
and again, D, E, and F,
then we have A to B as E to F,
and „ „ B to CO as D to E,
again „ A to CO as D to F.

AXIOMS.

1. Equi-multiples, or equi-submultiples of the same, or equal magnitudes, are equal.
2. Magnitudes of which the same is an equi-multiple, or an equi-submultiple, or of magnitudes of which equal magnitudes are equi-multiples or equi-submultiples, are also equal to each other.
3. A multiple or a sub-multiple of a greater magnitude is greater than an equi-multiple or equi-submultiple of a less.
4. A magnitude whose multiple or sub-multiple is greater than the equi-multiple or equi-submultiple of that other, is greater than that other.

PROPOSITION I. THEOREM.

If there be given two equal magnitudes (BC and DE), so often as any third magnitude (A) is contained in the one, so often it is contained in the other.

First let one of the given magnitudes, BC, be a multiple of A, and A is not oftener contained in the one than the other. For, if it be possible, let

A be contained oftener in DE than in BC; and as often as A is contained in BC, so often take away from DE, and let any part mE remain, therefore BC and Dm are equi-multiples of the same, A, and therefore BC is equal to Dm (by Ax. 1, B. 5), but BC is equal to DE (by Hypoth.), and therefore Dm is equal to DE (by Ax. 3, B. 1), which is absurd. Now, if it be possible, let A be contained oftener in BC than in DE, and take away A as often as possible from DE, letting a part mE remain less than A, take it away so often from BC, and since it is contained oftener in BC than in DE, a part nC remains greater than A, or equal to it, and therefore is greater than mE; but Bn and Dm are equi-multiples of the same A (by Const.), and therefore equal (by Ax. 1, B. 5), BC and DE are also equal (by Hypoth.), and therefore nC is equal to mE (by Ax. 3, B. 1), but also greater than it, which is absurd.

Secondly, let neither of the given magnitudes be a multiple of A, and A is not oftener contained in one than in the other.

For, if it be possible, let A be contained oftener in BC than in DE, and take it away from DE as often as it can be done, and mE shall remain less than A; take it away as often from BC, and nC will remain greater than A, and therefore greater than mE; but Bn and Dm are equi-multiples of the same A, and therefore equal (by Ax. 1, B. 5), and BC and DE are also equal (by Hypoth.), therefore nC is equal to mE (by Ax. 3, B. 1), but it is also greater than it, which is absurd.

So that in no case is A contained oftener in either of the given magnitudes, BC or DE, than in the other.

Cor. 1.—Hence it appears, if one of the given magnitudes be a multiple of A, the other will be an equi-multiple of A.

Cor. 2.—If two magnitudes be equal, as often as the first is contained in any third, so often is the second contained in the same third.

Cor. 3.—If there be four magnitudes, the first equal to the second, and the third equal to the fourth, as often as the first is contained in the third, so often is the second contained in the fourth.

PROPOSITION II. THEOREM.

If there be given two magnitudes (BC and DE), and if any assumed submultiple whatsoever of a third (A) be always contained so often in the one as in the other, the given magnitudes will be equal.

But, suppose one of them BC to be greater than the other, and let its excess be nC; and assume a submultiple a of A, less than nC, and a is contained oftener in BC than in Bn, and Bn is equal to DE, therefore as often as A is contained in Bn, so often is it contained in DE (by Prop. 1, B. 5), and therefore a is contained in BC oftener than in DE, but it is contained as often (by Hypoth.), which is absurd, therefore one of the given magnitudes is not greater than the other, and therefore they are equal.

Cor.—And if there be four magnitudes, of which the first is equal to the second, and any assumed submultiple of the first be always contained in the third as often as an equi-submultiple of the second is contained in the fourth, the third shall be equal to the fourth.

PROPOSITION III. THEOREM.

If there be four magnitudes (A, B, CD, and EF), of which the first is greater than the second, but the third equal to the fourth, the first is not contained oftener in the third than the second in the fourth.

First, let B be a submultiple of EF, and, if it be possible, let A be contained oftener in CD than B in EF; and let A be taken away from CD as often as B is contained in EF, and the part mD remains; and since Cm and EF are equi-multiples of A and B (by Const.), and A is greater than B (by

Hypoth.), Cm shall be greater than EF (by Ax. 3, B. 5), but EF is equal to CD (by Hypoth.), and therefore Cm is greater than CD, which is absurd. Now, let B be not a submultiple of EF, and, if it be possible, let A be contained oftener in CD than B in EF; take away B from EF as often as can be done, and nF remains less than B; and take away A as often from CD, and since A is contained oftener in CD than B in EF, the remainder mD shall be either greater than A or equal to it, and therefore greater than B, and also greater than nF; but Cm and En are equimultiples of A and B (by Const.), and A is greater than B, therefore Cm is greater than En (by Ax. 3, B. 5), but also mD is greater than nF, therefore the whole CD is greater than the whole EF, but it is also equal to it (by Hypoth.), which is absurd. So that A is in no case contained oftener in CD than B in EF.

Cor.—Similarly, if there be four given magnitudes, and the first is equal to the second, but the third less than the fourth, the first is not contained oftener in the third than the second in the fourth.

PROPOSITION IV. THEOREM.

If there be given two magnitudes (AB and CD), of which one (AB) is greater than the other (CD), there is given a submultiple of any third (E), which is contained oftener in the greater than in the less.

Let mB be the excess of AB above CD, and assume e a submultiple of E less than mB, and since Am is equal to CD, as often as e is contained in CD, so often it is contained also in Am (by Prop. 1, B. 5), but E is less than mB (by Const.), and therefore e is contained oftener in AB than in Am, and therefore oftener than in CD.

PROPOSITION V. THEOREM.

If there be two given magnitudes (AB and CD), and any third (e) can be assumed which is contained oftener in one (AB)

than in the other (CD), the latter (CD) shall be less than the former (AB).

If e be a submultiple of CD, take it away from AB as often as it is contained in CD, and let a part mB remain (by Hypoth.), and Am shall be equal to CD, because each is an equi-multiple of the same e, and therefore AB is greater than CD.

But if e be not a submultiple of CD, take it away as often as can be done, and nD remains less than e; and take it away as often from AB, and since it is contained oftener in AB than in CD (by Hypoth.), the remainder mB shall be greater than e, or equal to it, and therefore greater than nD; but Am and Cn are equi-multiples of e (by Const.), and therefore equal (by Ax. 1, B 5), but mB is greater than nD, therefore the whole AB is greater than CD.

PROPOSITION VI. THEOREM.

If there be given two magnitudes (a and AB), of which the first is a submultiple of the last, and there be assumed any third (x) which is a submultiple of the first (a), it will be also a submultiple of the last (AB).

For divide AB into two parts, Ao and oB, equal to a; and because x is a submultiple of a (by Hypoth.), it shall be also a submultiple of Ao (by Cor. 1, Prop. 1, B. 5), and similarly of oB, therefore divide Ao and oB into parts, Am, mo, on, and nB, equal to x, and the whole AB shall be divided into parts equal to x, and therefore x is a submultiple of AB.

PROPOSITION VII. THEOREM.

If there be two magnitudes (a and b) equi-multiples of two magnitudes (AC and BD), and there be assumed two others (x and z), which are equi-submultiples of the first, they shall be also equi-submultiples of the last.

For divide AC into parts Ao, oC, equal to a; and BD into parts, Bn, nD, equal to b; and since a and Ao are equal (by Const.), and x is a submultiple

of a (by Hypoth.), it shall be also a submultiple of Ao (by Cor. 1, Prop. 1, B. 5); and similarly, z is an equi-submultiple of b and Bn; but x and z are equi-submultiples of a and b (by Hypoth.), and therefore equi-submultiples of Ao and Bn; and so also equi-submultiples of oC and nD; whatever submultiple therefore x is of AC, such shall z be of BD.

Cor. 1.—If z be a submultiple of b, and be contained so often in b, as x is contained in an, and there be taken any equi-multiples, BD and AC, of b and an, z is not contained in BD oftener than x in AC.

For if x be a submultiple of an, z and x shall be equi-submultiples of BD and AC (by Prop. 7, B. 5), and therefore x is contained so often in AC, as z in BD. But if x be not a submultiple of an, take it away from an as often as possible, and let the part on remain; because x and z are equi-submultiples of ao and b (by Hypoth.), as often as z is contained in any multiple of b, so often is x contained in an equi-multiple of ao; but an is greater than ao, and therefore any multiple of an is greater than the equi-multiple of ao (by Ax. 3, B. 5); therefore x is not contained oftener in any multiple of ao than in an equi-multiple of an (by Cor. Prop. 3, B. 5), and therefore z is not contained oftener in any multiple of b, than x in an equi-multiple of an.

Cor. 2.—If b be a submultiple of BD, and it be contained as often in BD, as a is contained in AC, and z and x be taken equi-submultiples of b and a; z will not be oftener contained in BD than x in AC. For if a be a submultiple of AC, it appears that x and z are equi-submultiples of AC and BD (by Prop. 7, B. 5). But if not, take it away from AC as often as can be done, and let oC remain; therefore a and b are equi-submultiples of Ao and BD, and therefore since x and z are equi-submultiples of a and b (by

Hypoth.), as often as z is contained in BD, so often is x contained in Ao (by Prop. 7, B. 5), and therefore z is not oftener contained in BD, than x in AC.

COR. 3.—If z be a submultiple of BD, and z be contained in BD, as often as x is contained in the same BD; x shall be either equal to z, or less than it. If x be a submultiple of BD, it is evident that x and z are equal, since they are equi-submultiples of the same BD (by Ax. 1, B. 5). But if not, take away x from BD as often as z is contained in BD, and the part mD remains, therefore x and z are equi-submultiples of Bm and BD, and Bm is less than BD, and therefore x is less than z (by Ax. 4, B. 5).

PROPOSITION VIII. THEOREM.

If there be two unequal magnitudes (A and B), a certain submultiple of the less (B) is given, which is contained oftener in any third (CD), than an equi-submultiple of the greater (A) is contained in the same (CD).

For assume any sub-multiple, a of A, and an equi-submultiple, b of B; and let the first, a be a submultiple of CD, and as often as a is contained in CD, so often take away b from the same CD, and because b is less than a (by Hypoth. and Ax. 3, B. 5), a certain part mD remains; and if mD be greater than b, b is contained oftener in CD than in Cm, and therefore, oftener than a is contained in CD. But if mD be less than B, assume z a submultiple of b less than mD, and x an equi-submultiple of a; and since x and z are equi-submultiple of a and b, (by Const.), and a and b, equi-submultiples of CD and Cm, x and z shall be equi-submultiples of CD and Cm, (by Prop. 7, B. 5), but z is less than mD, (by Const.), and therefore, z is oftener contained in CD, than in Cm, and therefore, oftener than x is contained in the same CD.

Now, if a be not a submultiple of CD, and it be taken away as often as possible from CD, a certain part nD remains; and if b be taken away as often from CD, mD remains, and since Cm and Cn are equi-multiples of b and a, and b is less than a, (by Hypoth.), Cm will be less, than Cn (by Ax. 3, B. 5); assume z a submultiple of b, less than mn, and x an equi-submultiple of a, z and x, shall be equi-submultiples of Cm and Cn, (by Prop. 7, B. 5), but z is less than m and n, (by Const.), therefore, z is oftener contained in Cn than in Cm, and therefore, oftener than x in Cn; but x is greater than z, (by Const., and Ax. 3, B. 5), therefore, x is not contained oftener in nD, than z in the same nD, (by Prop. 3, B. 5), but z is contained oftener in Cn than x in Cn, therefore z is contained in CD oftener than x in CD.

Cor.—Similarly, if there be four magnitudes, of which the first, A, is greater than the second C, but the third, B, equal to the fourth, D; there can be assumed a sub-multiple of C, which is contained oftener in D, than an equi-submultiple of A in B.

PROPOSITION IX. THEOREM.

If there be two magnitudes (B and A), and there be given a submultiple (b) of one, which is contained oftener in any third (CD), than an equi-submultiple (a) of the other is contained in the same third, the latter (A) shall be greater than the former, (B).

Let the first b be a submultiple of CD, and add a as often as b is contained in CD, whence we make up EF; and because a is contained in EF, as often as b in CD, but b in CD oftener than a in CD, (by Hypoth.), a is contained oftener in EF than in CD; and therefore EF is greater than CD, (by Prop. 5, B. 5), but a and b, are equi-multiples of EF and CD, (by Const. and Hypoth.), therefore, A and B are equi-multiples of a and b, and therefore, A is greater than B (by Ax. 3, B. 5), since a is greater than b.

But if b be not a submultiple of CD, let it be taken away as often as possible, and let md remain, and let EF be assumed as above, and it shall be greater than CD, and therefore, greater than Cm, but a and b, are equi-multiples of EF and Cm, therefore, a is greater than b, and A and B being equi-multiples of a and b, therefore A is greater than B.

PROPOSITION X. THEOREM.

If there be four magnitudes (x, z, a and b), of which the first is contained oftener in the third than the second in the fourth, the first is also contained oftener in any multiple (AC) of the third, than the second in the equi-multiple (BD) of the fourth.

For divide AC into parts, Am and mC, equal to a; and BD into parts, Bn and nD equal to b; and since x is contained oftener in a than z in b (by Hypoth.), and x as often in Am as in a, and z as often in Bn as in b (by Prop. 1, B. 5), x is contained oftener in Am, than z in Bn; take away, *if possible*, z as often from BD, as x is contained in Am, and the residue oD shall be less than nD; therefore x is contained in Am as often as z in Bo (by Const.); but x is contained in mC oftener than z in nD (by Hypoth. and Prop. 1, B. 5), and therefore oftener than z in oD (by Cor. Prop. 3, B. 5), therefore x is contained oftener in AC than z in BD.

But if z cannot be taken away from BD as often as x is contained in Am, it appears that x is contained oftener in AC than z in BD.

Cor.—Hence if x be contained oftener in a than z in b, and a be contained oftener in AC than b in BD, x is also contained oftener in AC than z in BD.

PROPOSITION XI. THEOREM.

If two equal magnitudes (AC and BD) be divided into aliquot parts, unequal in number Am and mC; Bn, no and oD), the part (Bn) of those which are greater in number, shall be less than the part (Am) of those which are less in number.

For as often as Bn is contained in BD, so often add Am, and thence we have EF; because as often as Am is contained in EF, so often is Bn contained in BD (by Const.), but Bn is contained oftener in BD than Am in AC (by Hypoth.), Am is contained oftener in EF than in AC, therefore EF is greater than AC (by Prop. 5, B. 5), and therefore greater than BD, but Am and Bn are equi-submultiples of EF and BD (by Const.), therefore Bn is less than Am (by Ax. 3, B. 5).

PROPOSITION XII. THEOREM.

If there be any number of magnitudes (a and b), and the same number of equal magnitudes (AC and BD), each the equi-submultiple of each, whatever submultiple any one is of these, the same shall be the whole of all.

For since a is contained as often in AC as b in BD (by Hypoth.), there are as many parts in AC equal to a, as there are in BD equal to b: divide AC into parts, Am, mC, equal to a, and BD into parts, Bn, Dn, equal to b: therefore the number of parts, Am, mC, shall be equal to the number of parts BN, nD: and because Am is equal to a, and Bn to b, Am and Bn together shall be equal to a and b taken together. Similarly, nC and nD taken together are equal to a and b together; therefore there are as many parts in AC equal to a, as there are in AC and BD together, equal to a and b together; and therefore, such a sub-multiple is a of AC, such also is a together with b of AC together with BD.

Cor. 1.—If there be four magnitudes (a, b, AC, and BD), and as often as the first is contained in the third, so often is the second in the fourth, so often also is the first together with the second contained in the third together with the fourth. If a and b be submultiples of AC and BD, it is evident (by Prop. 12).

But if neither be a submultiple, take away a as often as possible from AC, and mC will remain less than a; and take away b as often from BD, and nD shall remain less than b; therefore a and b are equi-submultiples of Am and Bn (by Hypoth. and Const.), and therefore as often as a is contained in Am, so often also is a together with b contained in Am together with Bn (by Prop. 12), but a is greater than mC, and b is greater than nD, and therefore a and b together are greater than mC and nD together, therefore a together with b is not contained oftener in AC and BD together, than in Am and Bn together; and therefore as often as a together with b is contained in AC and BD together, so often is a contained in AC. It can be similarly demonstrated, if a be a submultiple of AC, and b not a submultiple of BD.

Cor. 2.—This is also true of any number of magnitudes.

PROPOSITION XIII. THEOREM.

If there be four magnitudes proportional, the first to the second, as the third to the fourth (A to CD, as B to EF), and there be assumed a submultiple (a) of the first, so that it be a submultiple of the second, an equi-submultiple (b) of the third shall be a submultiple of the fourth.

For it is not, but if it be possible, let a be a submultiple of CD, and b not a submultiple of EF; and as often as a is contained in CD, so often take away b from EF, and let aF remain: assume z a submultiple of b less than oF, and x an equi-submultiple of a, because x and z are equi-submultiples of a and b, and a and b equi-submultiples of CD and Eo, as often as x is contained in CD, so often is z contained in Eo, (by Prop. 7, B. 5), but oF is greater than z (by Const.), and therefore, z is contained oftener in EF than in Eo, and therefore, z is oftener contained in EF, than x in CD: but x and z are equi-submultiples of a and b (by Const.), and a and

b equi-submultiples of A and B, (by Hypoth.), therefore, x and z are equi-submultiples of A and B, (by Prop. 7, B. 5), and therefore, x is contained in CD as often as z in EF, (by Hypoth., and Def. 5, B. 5); but z was shown to be contained oftener in EF, than z in CD, which is absurd; there is not, therefore, any residue of EF, after b is taken from it as often as possible, therefore, b is a submultiple of EF.

PROPOSITION XIV. THEOREM.

Equal magnitudes (A and B) have the same ratio to the same magnitude (C). And (C) has the same ratio to the equals, (A and B).

PART 1.—Assume any equi-submultiples a and b, of the equal magnitudes A and B, and these shall also be equal, (by Ax. 1, B. 5), therefore, as often as a is contained in any third magnitude, C, so often is b contained in the same, C, (by Cor. 2, Prop. 1, B. 5), therefore, A is to C, as B to C, (by Def. 5, B. 1).

PART 2.—Assume any submultiple, c of C, and because A and B are equal, (by Hypoth.), as often as c is contained in A, so often also is it contained in B, (by Prop. 1. B. 5), therefore, C is to A, as C to B, (by Def. 5, B. 5).

COR.—If there be four magnitudes, A,B,C, and D, of which the first, A, is equal to the second B, and the third, C, equal to the fourth, D; A will be to C, as B to D.

PROPOSITION XV. THEOREM.

Magnitudes (A and B) which have the same ratio to the same magnitude, (C), are equal to one another. And magnitudes (A and B) to which the same magnitude (C) has the same ratio, are equal to one another.

PART 1.—For, if possible, let one of them, A, be less than the other, B, and there can be taken a submultiple of it, which is oftener contained in C, than an equi-submultiple of the greater B, is con-

tained in the same, C, (by Prop. 8, B. 5), contrary to the hypothesis, (by Def. 5, B. 5), therefore, the one is not less than the other, and therefore, they are equal.

Part 2.—For, if it be possible, let one B be greater than the other, and a submultiple of C can be taken, which is oftener contained in B, than in the greater A, (by Prop. 4, B. 5), contrary to hypothesis, (by Def. 5, B. 5), therefore, the one is not greater than the other, they are therefore equal.

PROPOSITION XVI. THEOREM.

Of unequal magnitudes (A and B) the less (A) has to the same (C) a less ratio than the greater (B).

And the same magnitude (C) has to the less (A) a greater ratio than to the greater (B).

Part. 1.—For since A is less than B, a submultiple of A can be assumed, which is contained oftener in C, than an equi-submultiple of B is contained in the same, C, (by Prop. 8, B. 5), and therefore, A has a greater ratio to C, than B to the same C, (by Def. 7, B. 5).

Part 2.—Because A is less than B, a submultiple of C can be taken, which is contained oftener in B than in A, (by Prop. 4, B. 5), therefore, C has a greater ratio to A than to B, (by Def. 7, B. 5).

PROPOSITION XVII. THEOREM.

Of magnitudes (A and B) which have a ratio to the same magnitude, (C), that which has the greater ratio (B) is the greater.

But that magnitude (A) to which the same (C) has a greater ratio, is the less.

Part 1.—Because the ratio which B has to C, is greater than the ratio of A, to the same C, a submultiple of A, can be taken, which is oftener contained in C, than an equi-submultiple of B in the same C, (by Def. 7, B. 5), and therefore, A is less than B, (by Prop.), B. 5.)

Part 2.—Since C has a greater ratio to A than to B, a submultiple of C can be taken, which is oftener contained in B than in A, (by Def. 7, B. 5), and therefore B is greater than A, (by Prop. 5, B. 5).

PROPOSITION XVIII. THEOREM.

Ratios which are the same, with the same ratio, are themselves the same.

If A is to B, as C to D, and C to D, as E to F, A will be also to B, as E to F.

Because A is to B, as C to D, as often as any submultiple of A is contained in B, so often is an equi-submultiple of C contained in D, (by Def. 5, B. 5); and because C is to D as E to F, as often as any submultiple of C is contained in D, so often is an equi-submultiple of E contained in F, (by Def. 5, B. 5), therefore, as often as any submultiple of A is contained in B, so often is an equi-submultiple of E contained in F, therefore A is to B as E is to F, (by Def. 5, B. 5).

PROPOSITION XIX. THEOREM.

If the first magnitude has the same ratio to the second, as the third to the fourth (A to B, as C to D), but the third has to the fourth a greater ratio than the fifth to the sixth, (C to D, than E to F), the first has also to the second a greater ratio than the fifth to the sixth, (A to B than E to F.)

Since C has to D a greater ratio than E to F, a submultiple of E can be taken which is contained oftener in F, than an equi-submultiple of C in D (by Def. 7, B. 5), and therefore, since A is to B as C to D, oftener than an equi-submultiple of A is contained in B, (by Def. 5, B. 5), therefore A has a greater ratio to B, than E to F, (by Def. 7, B. 5).

Cor.—Similarly, if A be to B, as C to D, but E to F in a greater ratio than C to D, E has also a greater ratio to F than A to B.

PROPOSITION XX. THEOREM.

If there be four magnitudes proportional, (A to B as C D to E), they will be also inversely proportional, (B to A, as E to CD).

For if there be taken any equi-submultiples, b and e of B and E, as often as b is contained in A, so often also is e contained in CD.

For if not, but if it be possible, let either of them, b, be oftener contained in A, than e is contained in CD; and as often as b is contained in A, so often let e be repeated, and make up CY, and CY will be greater than CD, (by Prop. 5, B. 5), and e is a submultiple of CY; let x be an equi-submultiple of CD, and also z of A. Therefore, because CD is less than CY, but x and e are equi-submultiples of them, x shall be less than e, (by Ax. 3, B. 5), therefore there is given a certain submultiple of x, which is contained oftener in E, than an equi-submultiple of e is contained in the same E, (by Prop. 8, B. 5), that is, than an equi-submultiple of b is contained in B, (by Hypoth. and Prop. 7, B. 5); but since z repeated as often as b is contained in A, makes A itself (by Const.), z is either greater than b or equal to it, (by Cor. 3, Prop. 7, B. 5); therefore, no submultiple of z is oftener contained in B, than an equi-submultiple of b is contained in the same B. (by Prop. 9, B. 5), but there is given a submultiple of x, which is oftener contained in E, than an equi-submultiple of b is contained in the same B, and therefore oftener than an equi-submultiple of z is contained in B, but x and z are equi-submultiples of CD and A, (by Prop. 8, B. 5), and therefore equi-submultiples of x and z, are also equi-submultiples of CD and A, (by Prop [illegible]) therefore there is

not given a submultiple of x, which is oftener contained in E, than an equi-submultiple of z is contained in B, (by Hypoth. and Def. 5, B. 5), but it is also given, which is absurd; therefore there is not given a submultiple of B which is oftener contained in A, than an equi-submultiple of E is contained in CD: it can be similarly demonstrated, that there is not a submultiple of E given, which is oftener contained in CD, than an equi-submultiple B in A: therefore B is to A, as E to CD, (by Def. 5, B. 5).

PROPOSITION XXI. THEOREM.

If there be four magnitudes proportional, (A to B as C to D), the first will be to the first together with the second, as the third to the third together with the fourth, (A to A, together with B, as C to C together with D).

For take a and c equi-submultiples of A and C; and as often as a is contained in B, so often also is c contained in D, (by Hypoth. and Def. 5, B. 5); therefore since a and c are equi-submultiples of A and C, and as often as a is contained in A, so often is c contained in C, but as often as a is contained in B, so often is c contained in D; a is contained in A and B together, as often as c in C and D together; therefore A is to A together with B, as C to C together with D, (by Def. 5, B. 5).

COR.—If A be to B, as C to D, A together with B, will be to B, as C together with D to D.

For since A is to B as C to D, B shall be to A as D to C, (by Prop. 20, B. 5), and therefore B to A together with B, as D to C, together with D, (by Prop. 21), therefore A together with B is to A, as C together with D to D, (by Prop. 20, B. 5).

PROPOSITION XXII. THEOREM.

If there be any number of magnitudes proportional, (A to B as C to D), as one of the antecedents will be to one con-

sequent, (A to B), so are all the antecedents taken together, to all the consequents together, namely (A together with C, to B together with D).

For if there be taken equi-submultiples, a and c of A and C, a and c together, will be the same submultiple of A and C together, (by Prop. 12, B. 5), but a is contained in B as often as c is contained in D, (by Hypoth. and Def. 5, B. 5), and therefore as often as a together with c, is contained in B together with D, (by Cor. 1, Prop. 12, B. 5), therefore A is to B, as A together with C, is to B together with D, (by Def. 5, B. 5).

PROPOSITION XXIII. THEOREM.

If there be four magnitudes proportional, (A to B as C to D), but the first be greater than the third, the second shall be greater than the fourth, and if equal, equal: and if less, less.

PART 1.—Let A be greater than C, and B will be greater than D. For if not, but if it be possible, let D be equal to B, and since A is greater than C, let B be equal to D, there can be taken a submultiple of C, which is oftener contained in D, than an equi-submultiple of A is contained in B, (by Cor. Prop. 8, B. 5), contrary to the hypothesis, (by Def. 5, B. 5). It can be similarly demonstrated, that D is not less than B.

Therefore B is greater than D.

PART 2.—If A be equal to C, B will be equal to D. For since A is to B as C to D, (by Hypoth.), as often as any submultiple of A is contained in B, so often is an equi-submultiple of C contained in D, (by Def. 5, B. 5), and therefore since A is equal to C, B will be equal to D, (by Cor. Prop. 2, B. 5).

PART 3.—If A be less than C, B will be less than D. For it can be demonstrated, as in the first part, that B is neither equal to, nor greater than D, therefore B is less than D.

COR.—And if the second be greater than the fourth, the first shall be greater than the third, and if equal, equal; and if less, less.

PROPOSITION XXIV. THEOREM.

If there be four magnitudes proportional, (A to B as C to D), but the first be greater than the second, the third will be greater than the fourth; and if equal, equal; and if less, less.

PART 1.—Let A be greater than B, and C will be greater than D. For since A is greater than B, there is a submultiple of A which is oftener contained in it than in B, (by Prop. 4, B. 5), and therefore an equi-submultiple of C is oftener contained in it than in D, (by Hypoth. and Def. 5, B. 5), therefore C is greater than D, (by Prop. 5, B. 5).

PART 2.—Let A be equal to B, and C will be also equal to D. For since A is equal to B, as often as a submultiple of A is contained in A, so often also is it contained in B, (by Prop. 1, B. 5); but as often as a submultiple of A is contained in B, so often is an equi-submultiple of C contained in D, (by Hypoth. and Def. 5, B. 5), and therefore as often as a submultiple of C is contained in C, so often also is it contained in D, therefore C is equal to D, (by Prop. 2, B. 5).

PART. 3.—Let A be less than B, and C will be less than D. For, since A is less than B, there is a submultiple of A which is oftener contained in B, than in A, (by Prop. 4, B. 5), and therefore an equi-submultiple of C is oftener contained in D than in C, (by Hypoth. and Def. 5, B. 5); therefore D is greater than C, (by Prop. 5, B. 5).

COR.—And if the third be greater than the fourth, the first will be greater than the second; and if equal, equal; and if less, less.

PROPOSITION XXV. THEOREM.

If there be four magnitudes proportional, (A to B as C to D,) but the first be less than the second, the first will be to the excess of the second above it, as the third to the excess of the fourth above it.

For take a and c equi-submultiples of A and C, and as often as a is contained in B, so often also is c contained in D, (by Hypoth. and Def. 5, B. 5); therefore since a and c are equi-submultiples of A and C, and as often as a is contained in B, so often also is c contained in D, a is contained as often in the excess of B above A, as c in the excess of D above C, therefore A is to the excess of B above A, as C to the excess of D above C, (by Def. 5, B. 5).

Cor. 1.—Similarly, if there be four magnitudes proportional, but the first be greater than the second, the first will be to its excess above the second, as the third to its excess above the fourth.

Cor. 2.—If there be four magnitudes proportional, A to B as C to D, the difference between the first and the second will be to the second; as the difference between the third and the fourth to the fourth.

For since A is to B as C to D, B will be to A as D to C, (by Prop. 20, B. 5), and therefore B is to the difference between itself and A, as D to the difference between itself and C, (by Cor. and Prop. 25, B. 5), therefore the difference between A and B is to B, as the difference between C and D to D (by Prop. 20, B. 5.)

PROPOSITION XXVI. THEOREM.

If there be four magnitudes proportional, (A to B as C to D), any submultiple (a) of the first will be to the second, as an equi-submultiple (c) of the third to the fourth.

For take x and z any equi-submultiples of a and c, and since x and z are equi-submultiples of A and C, but a and c are equi-submultiples of A and C (by Hypoth.) x and z shall be equi-submultiples of A

and C, (by Prop. 7, B. 5,) and therefore as often as x is contained in B, so often is z contained in D (by Hypoth. and Def. 5, B. 5) therefore a is to B, as c to D (by Def. 5, B. 5).

COR.—If there be two magnitudes, A and B, and there be taken any equi-submultiples of them, a and b, a shall be to A as b to B.

PROPOSITION XXVII. THEOREM.

If there be four magnitudes proportional, (A to B as C to D), the first shall be to any submultiple (b) of the second, as the third to an equi-submultiple (d) of the fourth.

For if not, but if it be possible, let A have a greater ratio to b than C to d; and since C has a less ratio to d than A to b, a submultiple z can be taken of C, which is oftener contained in d than an equi-submultiple x of A is contained in b, (by Def. 7, B. 5); therefore since z is oftener contained in d than x in b, but D and B are equi-submultiples of d and b, z is oftener contained in D than x in B, (by Prop. 10, B. 5,) but A is to B as C is to D, and therefore as often as x is contained in B, so often is z in D, (by Def. 5, B. 5,) which is absurd; therefore A has not a greater ratio to b than C to d, and similarly, C has not a greater ratio to d than A to b.

COR. 1.—If there be two magnitudes, A and B, and there be taken equi-submultiples of them, a and b; A will be to a as B to b.

COR. 2.—If there be four magnitudes proportional, A to B as C to D, and there be taken any equi-submultiples, a and c, of the antecedents, and also any other equi-submultiples, b and d, of the consequents, a shall be to b as c to d.

For since A is to B as C to D, and a and c are equi-submultiples of A and C, (by Hypoth.), a will be to B as c to D, (by Prop. 26, B. 5), and therefore since b and d are equi-submultiples of B and D, a shall be to b as c to d, (by Prop. 27, B. 5).

PROPOSITION XXVIII. THEOREM.

Parts (a and b) compared among themselves, have the same ratio, which their equi-multiples have, (AC and BD).

For since AC and BD are equi-multiples of a and b, there are as many parts in AC equal to a, as there are in BD equal to b; divide AC into the parts, Am, mn, nC, each equal to a, and BD into parts, Bo, os, sD, equal to b.

Therefore, Am is to Bo, as mn to os, (by Cor. Prop. 14, B. 5), and mn to os as nC to sD, (by same); therefore as one antecedent, Am, is to one consequent, Bo, so are all the antecedents together, or AC, to all the consequents together, or BD, (by Prop. 22, B. 5); but as Am is to Bo, so is a to b, (by Cor. Prop. 14, B. 5), therefore as a is to b, so is AC to BD, (by Prop. 18, B. 5).

Cor. 1.—If there be four magnitudes proportional, a to b as c to d, and there be taken equi-multiples, A, B, of a and b, and also other equi-multiples, C and D, of c and d, A shall be to B as C to D, as is evident from Prop. 28, and Prop. 18, B. 5.

Cor. 2.—Similarly, if A be to B as C to D, and there be taken equi-submultiples, a and b of A and B, and any equi-submultiples, c and d of C and D, a shall be to b as c to d.

PROPOSITION XXIX. THEOREM.

If there be four magnitudes proportional, (a to b as c to d) any multiple (A) of the first will be to the second, as an equi-multiple (C) of the third to the fourth.

For such a multiple as A is of a, let B be the same multiple of b, and also D of d, and A will be to B as C to D, (by Cor. 1, Prop. 28, B. 5); therefore A is to b as C to d, (by Prop. 27, B. 5).

PROPOSITION XXX. THEOREM.

If there be four magnitudes proportional (a to b as c to d) the first will be to any multiple (B) of the second, as the third to an equi-multiple (D) of the fourth.

For such a multiple as B is of b, let A be the same of a, and also C of c, A will be to B as C to D, (by Cor. 1, Prop. 28, B. 5), and therefore a is to B as c to D, (by Prop. 26, B. 5).

PROPOSITION XXXI. THEOREM.

If there be four magnitudes proportional (a to b as c to d), and there be taken equi-multiples (A and C) of the antecedents, and any other equi-multiples (B and D) of the consequents, these multiples will be also proportional.

For since a is to b as c to d, but A and C are equi-multiples of a and c, (by Hypoth.), A will be to b as C to d, (by Prop. 29, B. 5), therefore since B and D are equi-multiples of b and d, (by Hypoth.), A will be to B as C to D, (by Prop. 30, B. 5).

PROPOSITION XXXII. THEOREM.

If there be four magnitudes proportional (a to b as c to d), any equi-multiples (A and C) of the first and third, are together either greater, or equal, or less than the equi-multiples (B and D) of the second and fourth, however multiplied.

For since A and C are equi-multiples of the antecedents, (by Hypoth.), but B and D equi-multiples of the consequents, (by Hypoth.), A shall be to B as C to D, (by Prop. 31, B. 5); and therefore if A be greater than B, C will be greater than D; and if equal, equal; and if less, less (by Prop. 24, B. 5).

PROPOSITION XXXIII. THEOREM.

If there be four magnitudes of the same kind proportional (A to B as C to DO), they will be also alternately proportional, (A to C as B to DO).

For if there be taken any equi-submultiples a and b of A and B; as often as a is contained in C, so often is b contained in DO.

For if not, but if it be possible, let a be contained oftener in C than b in DO, and as often as a is contained in C, so often repeat b, and hence make up DY; DY shall be greater than DO (by Prop. 5. B. 5); therefore b is a submultiple of DY, let x be the equi-submultiple of DO, and z also of C.

Therefore, because x and b are equi-submultiples of DO and DY, (by Const.) but DO is less than DY, x shall be less than b, (by Ax. 3, B. 5), and since z and x are equi-submultiples of C and DO, (by Const.), z will be to x as C to DO, (by Prop. 28, B. 5); but C is to DO as A to B, (by Hypoth.), and therefore z is to x as A to B, (by Prop. 18, B 5); but a and b are equi-submultiples of A and B, and therefore a is to b as A to B, (by Prop. 28, B. 5), therefore z is to x as a to b, (by Prop. 18, B. 5), but x is less than b, and therefore z is less than a (by Prop. 23, B. 5), therefore there is given a submultiple of z, which is oftener contained in C, than an equi-submultiple of a is contained in the same C, (by Prop. 8, B. 5); but z and x are equi-submultiples of C and DO, (by Const.), and therefore as often as any submultiple of z is contained in C, so often is an equi-submultiple of x contained in DO, (by Prop. 7, B. 5); therefore there is a submultiple of x, which is oftener contained in DO, than an equi-submultiple of a is contained in C; but x repeated as often as a is contained in C, makes up DO, (by Const.); therefore there is not a submultiple of x which is oftener contained in DO, than an equi-submultiple of a is contained in C, (by Cor. 2, Prop. 7, B. 5), but it is also given, which is absurd.

Therefore there is not any submultiple of A contained in C, oftener than an equi-submultiple of B is contained in DO ; and it can be similarly demonstrated that there is no submultiple of B oftener contained in DO, than an equi-submultiple of A is contained in B; therefore A is to C as B to DO, (by Def. 5, B. 5).

PROPOSITION XXXIV. THEOREM.

If there be any number of magnitudes, and an equal number of others, which if taken two by two are in the same ratio, and if their proportion be ordinate; they shall be ex æquali in the same ratio.

First, let there be three magnitudes, A, B, and CO, and three others D, E, and F; if A be to B as D to E, and B to CO as E to F, A shall be to CO as D to F. For if there be taken any equi-submultiples, a and d, of A and D, as often as a is contained in CO, so often is d contained in F.

For if not, if it be possible, let d be contained oftener in F than a in Co; and as often as d is contained in F, so often repeat a, and hence make up CY; CY shall be greater than CO, (by Prop. 5, B. 5); a is therefore a submultiple of CY, let x be the equi-submultiple of CO, as also z of F.

Therefore since x and a are equi-submultiples of CO and CY, (by Const.), but CO is less than CY, x will be less than a, (by Ax. 3, B. 5), and since B is to CO as E to F, (by Hypoth.), CO shall be to B as F to E; but x and z are equi-submultiples of CO and F, (by Const.), and therefore x is to B as z to E, (by Prop. 26, B. 5); therefore as often as any submultiple of x is contained in B, so often is an equi-submultiple of z contained in E; but x is less than a, and therefore there is given a submultiple of x, which is oftener contained in B, than an equi-submultiple of a is contained in the same B, (by Prop. 8, B. 5) ; and therefore oftener than an equi-submultiple of d is contained in E, (by Hypoth. and Def. 5, B. 5); therefore there

is a submultiple of z given, which is oftener contained in E, than an equi-submultiple of d is contained in E, therefore z is less than d, (by Prop. 9, B. 5); but z repeated as often as d is contained in F, makes up F, (by Const.); and therefore z is either greater than d or equal to it, (by Cor. 3, Prop. 7, B. 5), but it was shown to be less than it, which is absurd.

Therefore there is not any submultiple of D contained oftener in F, than an equi-submultiple of A is contained in CO; and it can be similarly demonstrated that there is no submultiple of A contained often in CO, than an equi-submultiple of D is contained in F: therefore A is to C as D to F, (by Def. 5, B. 5).

Now let there be four magnitudes, A, B, C, G, and four others, D, E, F, H; and let A be to B, as D to E, and B to C as E to F, and C to G as F to A, A shall be also to G as D to H.

For since there are three magnitudes, A, B, C, and three others, D, E, F, which if taken two by two, are in the same ratio, A will be to C as D to F, (by preceding part), but C is to G as F to H, (by Hypoth.), and therefore A shall be to G as D to H (by preceding part); and similarly, whatever be the number of magnitudes given.

PROPOSITION XXXV. THEOREM.

If there be three magnitudes (A, B, C), and three others (D, E, F), which if taken two by two are proportional, but perturbate (A to B as E to F, and B to C as D to E), and if in the first series, the first magnitude be greater than the third, the first in the next series shall be also greater than the third.

For since D is to E as B to C (by Hypoth.), E will be to D as C to B (by Prop. 20, B. 5), but A is greater than C (by Hypoth.), therefore A has a greater ratio to B, than C has to the same B, (by Prop. 16, B. 5), and therefore a greater ratio than E has to D (by Cor. Prop. 19, B. 5): but A is to

B as E to F (by Hypoth.), and therefore E has a greater ratio to F, than the same E has to D (by Prop. 19, B. 5), therefore D is greater than F (by Prop. 17, B. 5).

PROPOSITION XXXVI. THEOREM.

If there be three magnitudes (A, B, C), and three others (D, E, F), which, if taken two by two are proportional, but perturbate (A to B as E to F, and B to C as D to E), and if there be taken any equi-submultiples (a, b, and d) of the first and second in the first series, and of the first in the last series (of A, B, and D); and also any equi-submultiples (c, e, and f): of the third in the first series, and of the second and third in the next series (of C, E, and F), these equi-submultiples of the quantities in the first series, and of these in the second series, shall be also perturbate proportionals (a to b as e to f, and b to c as d to e).

For since A is to B as E is to F (by Hypoth.), and a and b are taken equi-submultiples of A and B, and e and f equi-submultiples, of E and F, a will be to b as e to f (by Cor. 2, Prop. 28, B. 5).

And since B is to C as D to E (by Hypoth.), and b and d are equi-submultiples of B and D, and also c and e equi-submultiples of C and E (by Hypoth.), b will be to c as d to e (by Cor. 2, Prop. 27, B. 5).

PROPOSITION XXXVII. THEOREM.

If there be three magnitudes (A, B, C), and three others (D, E, F), which, if taken two by two are proportional, but perturbate (A to B as E to F, and B to C as D to E): and if there be taken any equi-submultiples (a and d) of the first in each series (A and D), and any other equi-submultiples (c and f) of the last in each series (C and F), and if the part (a) of the first in the first series, be greater than the part (c) of the last in the same series, the part (d) of the first in the second series will be greater than the part (f) of the last.

For take b, which is the same submultiple of B that a is of A, and also e, which is the same submultiple of E that f is of F; there will be three magnitudes, a, b, c, and three others, d, e, f, if taken two by two are proportional, but perturbate (by Prop. 36, B. 5); and therefore if a be greater than e, d will be greater than f (by Prop. 35, B. 5).

PROPOSITION XXXVIII. THEOREM.

If there be any number of magnitudes, and an equal number of others, which, if taken two by two are in the same ratio, but perturbate; they will be also ex æquali in the same ratio.

First, let there be three magnitudes, A, B, and CO, and three others, D, E, F; if A be to B as E to F, and B to CO as D to E, A will be to CO as D to F. For if there be taken any equi-submultiples, a and d, of A and D, as often as a is contained in CO, so often is d contained also in F. For if not, but, if it be possible, let d be contained oftener in F than a in CO; and as often as d is contained in F, so often repeat a; and hence let CY be made up; CY shall be greater than CO (by Prop. 5, B. 5). Therefore a is a submultiple of CY; let x be an equi-submultiple of CO, and also z of F. Therefore since x and a are equi-submultiples of CO and CY, but CO is less than CY, x shall be less than a (by Ax. 3, B. 5); and since z is an equi-submultiple of F, and x of CO (by Const.), but as often as d is contained in F, so often is x in CO (by Const.) d is contained in F as often as z, a submultiple of F, is contained in the same F, therefore d is either less than z or equal to it (by Cor. 3, Prop. 7, B. 5). Therefore there are taken equi-submultiples, a and d, of A and D, and also other equi-submultiples, x and z of CO and F, and a is greater than x, but d is not greater than z, which is impossible (by Prop. 37, B. 5). Therefore no submultiple of D is contained oftener in

F, than an equi-submultiple of A is contained in CO; and it can be similarly demonstrated that no submultiple of A is contained oftener in CO, than an equi-submultiple of D is contained in F, therefore A is to CO as D to F.

Now let there be four magnitudes, A, B, C, G, and four others, D, E, F, H; and let A be to B as F to H; but B to C as E to F, and C to G as D to E; A will be also to G, as D to H. For since there are three magnitudes, A, B, C, and three others, E, F, H, which, if taken two by two are proportional, but perturbate, A will be to C as E to H (by part preceding), but C is to G as D to E (by Hypoth.), therefore A will be to G, as D to H (by part preceding). And similarly, if there be any number of magnitudes given.

PROPOSITION XXXIX. THEOREM.

If there be three magnitudes proportional, (A to B as B to C), and three others proportional, (D to E as E to F); and there be, in the first series, the first to the last, (A to C), as the first to the last, (D to F), in the next series; the first will be also to the second, in the first series, (A to B), as the first to the second in the next series, (D to E).

For if not, but if it be possible, let one of them, A, have a less ratio to B, than D has to E; and since B is to C as A to B (by Hypoth.), and E to F as D to E (by Hypoth.), B also has a less ratio to C, than E has to F, (by Prop. 19, B. 5); therefore a submultiple of B can be taken, which is oftener contained in C, than an equi-submultiple of E is contained in F, (by Def. 7, B. 5): let them be b and e; and take equi-submultiples, a and d, of A and D, and since they are assumed equi-submultiples, a and b of A and B, a will be to b as A to B, (by Prop. 28, B. 5), and similarly d is to e as D to E, (by same), therefore a has a less ratio to b than d has to e, (by Hypoth.), and therefore there can be taken

a submultiple of a, which is contained oftener in b, than an equi-submultiple of d is contained in e, (by Def. 7, B. 5); let them be also assumed x and z; and since x is contained oftener in b than z in d, but b is contained oftener in C than d in F, x is also contained oftener in C than z in F, (by Cor. Prop. 10, B. 5), but x and z are equi-submultiples of a and d, and a and d are equi-submultiples of A and D, and therefore x and z are equi-submultiples of A and D, (by Prop. 7, B. 5); therefore as often as x is contained in C, so often also is z contained in F, (by Hypoth. and Def. 5, B. 5); but x was shown to be oftener contained in C, than z in F, which is absurd. Therefore A has not a less ratio to B, than D has to E; and it can be similarly demonstrated, that D has not a less ratio to E, than A has to B; therefore A is to B, as D to E.

END OF THE FIFTH BOOK.

SIXTH BOOK.

DEFINITIONS.

1. Similar rectilineal figures, are those which have the angles in each respectively equal, and the sides about the equal angles proportional.

2. A right line is said to be cut in extreme and mean ratio, when, as the whole is to the greater segment, so is the greater to the less.

3. The altitude of any figure, is a right line drawn from the vertex perpendicular to the base.

4. A parallelogram described upon a right line, is said to be applied to that line.

5. A parallelogram described upon any part of any right line, is said to be applied to that line *deficient by a parallelogram;* namely, by that parallelogram which is described upon the remaining part.

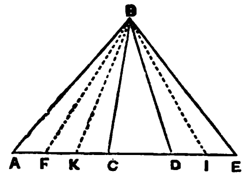
6. When a given right line is produced, the parallelogram described upon the whole is said to be applied to the given right line, *exceeding by a parallelogram;* namely, by that parallelogram which is described upon the produced part.

PROPOSITION I. THEOREM.

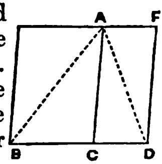
Triangles (ABC, DBE) and parallelograms, (BA and CF), which have the same altitude, are to one *another as their* bases.

PART 1. Let the base of the triangle ABC be divided into any number of equal parts, AF, FK, and KC, and

take on the base DE, as often as possible, the parts, DI and IE, equal to AF, and draw BF, BK, and BI.



Therefore, since the right lines AF, FK, KC, DI, and IE are equal (by Const.) but the triangles constructed upon them are of the same altitude, they shall be also equal (by Prop. 38, B. 1.); therefore such a submultiple as AF is of AC, such is the triangle ABF of ABC, and as often as AF is contained in the base DE, so often is the triangle ABF contained in the triangle DBE, and it can be similarly shown that as often as any other submultiple of AC is contained in DE, so often is an equi-submultiple of ABC contained in DBE; therefore the triangle ABC is to the triangle DBE, as the base AC to the base DE (by Def. 5, B. 5).

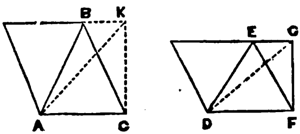


Part 2.—Parallelograms, BA and CF, of the same altitude, are to one another as their bases, BC and CD. For draw BA and AD, and since the triangles BAC, CAD, are of the same altitude, they shall be to one another as their bases BC and CD (by preceding part); but the parallelograms are double of them (by Prop. 34, B. 1), and therefore the parallelograms are to one another as their bases BC and CD (by Prop. 28, B. 5).

Cor. 1.—Triangles or parallelograms which have equal altitudes, are to one another as their bases. For, placing the bases in directum, a right line joining the vertices will be parallel to the right line in which the bases are; for letting perpendiculars fall from the vertices on the base, these shall be parallel; and since they are also equal (by Hypoth.), the right lines joining them will be parallel (by Prop. 33, B. 1); and therefore, it can be similarly demonstrated as in the

Proposition, that the triangles or parallelograms are to one another as their bases.

Cor. 2.—Triangles, ABC, DEF, and parallelograms, AB and DE, upon equal bases AC and DF, are to one another as their altitudes.



If the triangles be rectangular, it is evident. If not, draw BK and EG parallel to AC and DF, and through C and F draw CK and FG perpendicular to AC and DF, and join AK and DG.

Therefore since the triangles ABC and AKC are upon the same base, and between the same parallels, ABC shall be equal to AKC (by Prop. 37, B. 1); and DGF is similarly equal to DEF; but if the right lines CK and FG be assumed for the bases of the triangles AKC and DGF, their altitudes AC and DF shall be equal (by Hypoth.), and therefore, they are to each other as their bases CK and FG (by preceding Cor.), therefore ABC and DEF, which are equal to them, are to one another as CK and FG, but these lines are equal to their altitudes (by Prop. 34, B. 1, Def. 3. B. 6), but the parallelograms are double of the triangles, and therefore are also to each other as CK and FG.

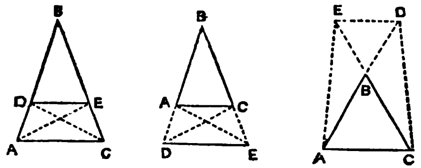
PROPOSITION II. THEOREM.

If a right line (DE) be drawn parallel to any side (AC) of a triangle (ABC), it shall cut the remaining sides, or the sides produced proportionally.

And if the right line (DE) cut the sides of the triangle, or the sides produced, proportionally; it shall be parallel to the remaining side (AC).

Part 1.—Let DE be parallel to AC, and AD will be to DB as CE to EB. For draw AE and DC, and since the triangles EAD and ECD are upon the same base ED, and between the same parallels ED and CA

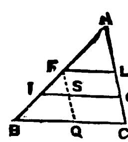
they will be equal (by Prop. 37, B. 1); therefore AED



has the same ratio to DEB, which CDE has to the same EDB (by Prop. 14, B. 5); but AED is to DEB, as AD to DB (by Prop. 1, B. 6); and CDE is to EDB, as CE to EB (by Prop. 1, B. 6); therefore AD has the same ratio to DB, that CE has to EB (by Prop. 18, B. 5).

Part 2.—Let AD be to DB, as CE to EB; and DE will be parallel to AC. For the preceding construction remaining, since AD is to DB as CE to EB; but as AD is to DB, so is the triangle AED to the triangle DEB (by Prop. 1, B. 6); and as CE is to EB, so is the triangle CDE to the triangle EDB (by same Prop.); AED will be to DEB as CDE to the same EDB (by Prop. 18, B. 5); therefore AED is equal to CDE (by Prop. 15, B. 5); but they are upon the same base DE, and on the same side of it, therefore DE is parallel to AC (by Prop. 39, B. 1).

Schol.—In the second part it is understood that the homologous segments are at the same side of the right line DE.

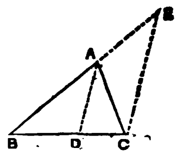


Cor.—If there be drawn several right lines, IO and FL, parallel to the side BC of the triangle BAC all the segments of the sides will be proportional. For draw FQ parallel to AC, in the triangle BFQ, BI will be to IF as QS to SF (by Prop. 2, B. 6), but on account of the parallelograms QO and SL, CO is equal to QS, and OL to SF (by Prop. 34, B. 1), and therefore CO is to OL as BI to IF; it can be similarly demonstrated, whatever be the number of parallels.

PROPOSITION III. THEOREM.

A right line (AD) bisecting the angle of a triangle (BAC), cuts the opposite sides into segments (BD, DC), proportional to the remaining sides (BA, AC).

And a right line (AD), drawn from the angle of a triangle, cutting the opposite side (BC) into segments (BD, DC) proportional to the conterminous sides (BA, AC), bisects that angle.



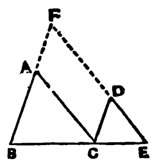
Part 1.—Through C draw CE, parallel to AD, and let BA produced meet CE in E. Therefore since AD and EC are parallel, the angle BAD is equal to the internal angle at the same side AEC (by Prop. 29, B. 1); and therefore the angle DAC is equal to AEC (by Hypoth. and Ax. 1, B. 1), but DAC is equal to the alternate angle ACE (by Prop. 29, B. 1); and therefore AEC and ACE are equal, therefore the opposite sides AC and AE are equal (by Prop. 6, B. 1); but on account of the parallels, AD and EC, EA is to AB, as CD to DB (by Prop. 2, B. 6); therefore since EA and AC are equal, AC is to AB as CD to DB.

Part 2.—Letting the preceding construction remain, BA is to AE, as BD to DC (by Prop. 2, B. 6); but as BD to DC, so is BA to AC (by Hypoth.); and therefore BA is to AE, as BA to AC (by Prop. 18, B. 5), therefore AE and AC are equal (by Prop. 15, B. 5), and therefore the angle ACE is equal to the angle AEC (by Prop. 5, B. 1); but on account o the parallels AD and EC, the angle DAC is equal to the alternate angle ACE (by Prop. 29, B. 1), and the angle BAD is equal to the internal angle at the me side AEC; and, since ACE and AEC are equal, AC and BAD are also equal; therefore the right line D bisects the angle BAC.

Cor.—If a right line bisecting the vertical angle o a triangle bisects the base, it will be an isosceles triangle.

PROPOSITION IV. THEOREM.

The sides about the equal angles of equi-angular triangles (BAC and CDE) are proportional. But the sides which subtend the equal angles are homologous.

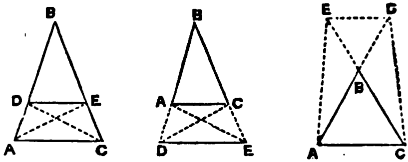


For let the sides BC and CE, opposite to the equal angles BAC and CDE, be placed in directum, so that the triangles may be at the same side, and that equal angles BCA and CED be not conterminous; and, since the angles ABC and BCA are less than two right angles (by Prop. 17, B. 1), but CED is equal to BCA ABC and CED will be less than two right angles, and therefore the right lines BA and ED must meet if produced (by Ax. 12, B. 1); therefore let them meet in F, and because the angles BCA and CED are equal (by Hypoth.), CA and EF shall be parallel (by Prop. 28, B. 1); and because ABC and DCE are equal, CD and BF will be also parallel (by same); therefore AFDC is a parallelogram, and therefore AC is equal to FD, and FA to CD (by Prop. 34, B. 1).

Therefore, since in the triangle BFE, AC is parallel to FE, BA will be to AF, or to CD equal to AF, as BC to CE (by Prop. 2, B. 6); and therefore, by alternation, AB is to BC as DC to CE; and, since CD is parallel to BF, BC shall be to CE as FD, or AC equal to FD, is to DE (by Prop. 2, B. 6); and therefore, by alternation, BC is to CA as CE to ED; therefore, since AB is to BC as DC to CE, and BC to CA as CE to ED, ex æquali (by Prop. 34, B. 5), AB will be to AC as DC to DE: therefore the sides about the equal angles are proportional, and those opposite to the equal angles homologous.

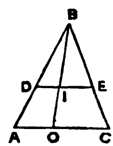
Schol.—It is manifest that the triangles BAC and CDE are similar; and also that the sides which are opposite to the equal angles are proportional.

COR. 1.—If in a triangle ABC a line DE be drawn parallel to any side AC, the triangle cut off, DBE, will be similar to the whole.



For on account of the common angle B, and the angles BDE, BED being equal to the internal angles at the same side, BAC, BCA (by Prop. 29, B. 1), it is equiangular, and therefore similar (by Schol.)

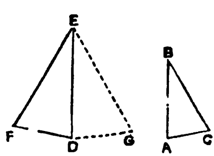
COR. 2.—If there be drawn a parallel DE to any side AC of the triangle ABC, the right line BO drawn from the opposite angle shall cut the parallels into proportional parts.



For since in the triangle ABO, DI is parallel to AO, AO will be to DI as OB to IB (by the Schol. of this Prop.); and similarly, OC is to IE as OB to IB; and therefore, AO is to DI as OC to IE (by Prop. 18, B. 5), and by alternation, AO is to OC as DI to IE.

PROPOSITION V. THEOREM.

If two triangles (ABC, DEF) have the sides proportional (BA to AC as ED to DF, and AC to CB as DF to FE), they will be equiangular; and the equal angles are subtended by homologous sides.

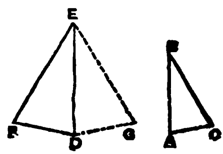


At the extremities of any side DE of either triangle DEF, let the angles EDG and DEG be constructed, equal to the angles A and B, at the extremities of the side AB, which is homologous to ED; and the remaining angle

G in the triangle DEG will be equal to the remaining angle C in the triangle ABC (by Cor. 2, Prop. 32, B. 1). Therefore, since the triangle ABC is equiangular to the triangle DEG (by Const.), BA will be to AC as ED to DG (by Prop. 4, B. 6); but BA is to AC as ED to DF (by Hypoth.), and therefore ED is to DG as ED is to DF (by Prop. 18, B. 5), therefore DG and DF are equal (by Prop. 15, B. 5); it can be similarly demonstrated that EG and EF are equal, therefore the triangle EDG is equilateral to the triangle EDF, and therefore equiangular (by Schol. Prop. 8, B. 1); but EDG is equiangular to BAC (by Const.), and therefore BAC is also equiangular to EDF; it is evident that the equal angles are subtended by the homologous sides.

PROPOSITION VI. THEOREM.

If two triangles (ABC and DEF) have one angle of the one equal to one angle of the other, (A to D) and the sides about the equal angles proportional (BA to AC as ED to DF), the triangles will be equiangular; and will have the angles equal, by which the homologous sides are subtended.



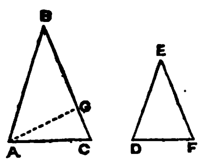
With either leg, DE, from the equal angles of either, A and D, and at either of its extremities, D, construct the angle EDG, equal to the angle A, and at the extremity E, construct the angle DEG equal to B, and the remaining angle G in the triangle DEG, will be equal to the angle C, in the triangle ABC (by Cor. 2, Prop. 32, B. 1).

Therefore since ABC is equiangular to the triangle DEG (by Const.), BA shall be to AC as ED to DG (by Prop. 4, B. 6); but BA is to AC as ED to DF (by Hypoth.); and therefore ED is to DG as ED to DF (by Prop. 18, B. 5), therefore DG and DF are equal

(by Prop. 15, B. 5); but the angles EDG and EDF are equal, since either is equal to the angle A, (by Hypoth. and Const.), and the side ED is common to both triangles, therefore the triangle EDF is equiangular to the triangle EDG (by Prop. 4, B. 1), but BAC is equiangular to EDG (by Const.), therefore BAC is also equiangular to EDF; but it is evident that the equal angles are subtended by the homologous sides.

PROPOSITION VII. THEOREM.

If two triangles (ABC, DEF), have one angle of the one equal to one angle of the other (B to E), and the sides about two of the remaining angles proportional (BA to AC, as ED to DF), but the two remaining angles (C and F), be either each less, or each not less than a right angle, the triangles will be equiangular, and shall have the angles equal, about which the sides are proportional.



First, let both angles, C and F, be less than a right angle; and BAC shall be equal to EDF.

For if not, but if it be possible, let either of them, BAC, be greater; and at the point A, with the right line BA, construct the angle BAG, equal to the less EDF.

For, since in the triangles, DEF, ABG, the angles E and B are equal (by Hypoth.), and EDF and BAG are also equal (by Const.), EFD will be equal to BGA (by Cor. 2, Prop. 32, B. 1); therefore the triangles are equiangular, and BA is to AG as ED to DF (by Prop. 4, B. 6), but BA is to AC, as ED to DF (by Hypoth.), and therefore BA is to AG as BA to AC (by Prop. 18, B. 5); therefore AG is equal to AC (by Prop. 15, B. 5); therefore the angle ACG is equal to the angle AGC (by Prop. 5, B. 1), and therefore each is acute (by Cor. Prop. 17, B. 1), and since AGC is acute,

AGB will be obtuse, and EFD equal to AGB, is also obtuse, but it is said to be acute, which is absurd.

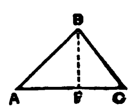
Therefore the angle BAC is not greater than EDF, and it can be similarly demonstrated that EDF is not greater than BAC, they are therefore equal; and since the angles ABC and DEF are also equal (by Hypoth.) the triangles shall be equiangular (by Cor. 2, Prop. 32, B. 1). therefore they have the sides about the equal angles, proportional (by Prop. 4, B. 6). It can be similarly proved if each angle, C and F, be not less than a right angle, that BAC and EDF are equal, and therefore, that the triangles are equiangular, and therefore that they have the sides about the equal angles proportional.

Cor. 1.—If in two triangles ABC and DEF, an angle B of the one be equal to the angle E of the other, and the sides about another two angles proportional, BA to AC, as ED to DF, and the remaining angle C be right, the triangles will be equiangular; for letting the construction of the preceding Proposition remain, it can be demonstrated that the triangle CAG is isosceles, and therefore that the angle AGC is equal to the right angle C, which is absurd.

Cor. 2.—If the sides opposite to any of the equal angles be equal, the triangles themselves will be equal, as appears from Prop. 26, B. 1.

PROPOSITION VIII. THEOREM.

If in a right angled triangle (ABC), a perpendicular (BF) be drawn from the right angle to the opposite side; it divides the triangle into parts both similar to the whole triangle and to each other.



For in the triangles ABF and ABC, the angle AFB is equal to the angle ABC (by Hypoth.), and the angle A is common to both, therefore the remaining angle ABF is equal to the remaining angle C (by Cor. 2, Prop. 32, B. 1), and therefore

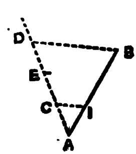
the triangles are equiangular: therefore the sides about the equal angles are proportional (by Prop. 4, B. 6), and the triangles are similar (by Def. 1, B. 6).

It can be similarly demonstrated that the triangle FBC is similar to the triangle ABC. Therefore, because the angle ABF is equal to the angle C, and the angles AFB and BFC are also equal, the remaining angle A shall be equal to the remaining angle FBC, therefore the triangles ABF and FBC are equiangular; and therefore the sides about the equal angles are porportional (by Prop. 4, B. 6), therefore the triangles are similar.

Cor.—Hence it appears that the perpendicular BF is a mean proportional between the segments AF and FC, of the side on which it falls; and that the remaining sides, AB and BC, are mean proportionals between the conterminous segments AF and FC, and the whole side AC.

PROPOSITION IX. PROBLEM.

From a given right line (AB) to cut off a required part.

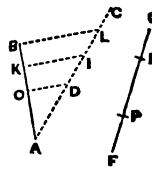


From either extremity, A, of the given line, draw AD, making any angle with AB, and take any point in it, C; make AD the same multiple of AC, that AB is of the part to be cut off, and draw BD, draw through C a right line, CI parallel to BD, AI will be the part required.

For AI is to AB, as AC to AD (by Cor. 1, Prop. 4, B. 6,) and therefore because AC is a submultiple of AD (by Const.), AI will be an equi-submultiple of AB (by Prop. 13, B. 5).

PROPOSITION X. PROBLEM.

To divide a given right line (AB) similarly to a given divided ..ne (FG).

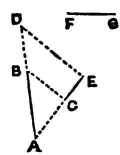


From either extremity A of the given divided line AB draw a right line AC, making any angle with it, and take AD, DI, and IL, equal to the parts FP, PR, and RG into which the given line FG is divided (by Prop. 3, B. 1), and draw LB; draw, through I and D, the right lines IK and DO parallel to LB.

Therefore, because in the triangle BAL, the right lines KI and OD are parallel to the side BL, BK will be to KO as LI to ID (by Cor. Prop. 2, B. 6), or as GR to RP (by Const.), and KO is to OA as ID to DA (by Cor. Prop. 2, B. 6), or as RP to PF (by Const.), and therefore the given line AB is divided similarly to FG.

PROPOSITION XI. PROBLEM.

To find a third proportional to two given right lines (AB and FG).

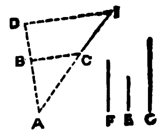


From either extremity of the given line AB draw the right line AE, making any angle with it, and take AC equal to the other given line FG, draw BC; and produce AB, in it take BD equal to FG; and through D draw DE parallel to BC, CE will be the third proportional required.

For since in the triangle DAE, the right line BC is parallel to the side DE, AB shall be to BD as AC to CE (by Prop. 2, B. 6), but BD and AC are equal to FG (by Const.), therefore AB is to FG as FG to EC.

PROPOSITION XII. PROBLEM.

To find a fourth proportional to three given right lines (F, E, and G).

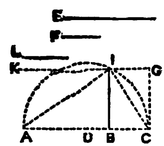


Draw two right lines AD and AI, containing any angle, and in them take AB, BD, and AC, equal to F, E, and G; and draw BC, draw through D the right line DI, parallel to BC; CI will be the fourth proportional required.

For since in the triangle DAI, the right line BC is parallel to the side DI, AB will be to BD as AC to CI (by Prop. 2, B. 6), but the given lines F, E, and G are equal to AB, BD, and AC (by Const.), therefore F is to E as G to CI.

PROPOSITION XIII. PROBLEM.

To find a mean proportional between two given right lines (E and F).



Draw any right line AC, take in it AB and BC equal to E and F: bisect AC in D, and from the centre D, through A, describe a semicircle AIC; and through B draw BI perpendicular to AC, meeting the circumference in I, BI will be the mean proportional required.

For draw AI and IC, and because in the triangle AIC, the angle I in a semicircle is a right angle (by Prop. 31, B. 3), but IB is a perpendicular drawn from it to the opposite side; BI shall be a mean proportional between AB and BC (by Cor. Prop. 8, B. 6), and therefore between E and F, which are equal to AB and BC (by Const.).

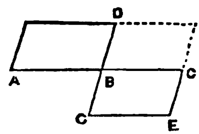
Cor. 1.—Mean proportionals can be similarly found between the given lines and this mean, whence arises a series of five right lines continually proportional; and again finding means between the adjacent terms in this series, a series of nine proportionals appears; and so on.

Cor. 2.—Having the sum of the extremes of three proportionals AC, and the mean L, the means them-

selves can be found; for describe a semicircle AIC, draw CG perpendicular to AC, and equal to the mean proportional L and draw GK parallel to AC; from the point I, in which it meets the circle, let fall a perpendicular IB upon AC; and this divides AC into parts AB, BC, which are the extremes required. For draw AI and IC, the angle AIC is right, and therefore IB is a mean proportional between AB and BC (by Cor. Prop. 8, B. 6), and IB is equal to CG, and therefore equal to the given line L.

PROPOSITION XIV. THEOREM.

Of equal parallelograms (AD and GC), which have also an angle of one equal to an angle of the other, the sides about the equal angles are reciprocally proportional (AB to BC as BG to BD). And if two parallelograms (AD and GC) have an angle in the one equal to an angle in the other, and the sides about the equal angles reciprocally proportional; they shall be equal to each other.



PART 1.—Let the two sides, AB and BC, about the equal angles be placed in directum, and because ABD and DBC are equal to two right angles (by Prop. 13, B. 1), and GBC is equal to ABD (by Hypoth.), GBC and DBC shall be equal to two right angles, and therefore GB and DB are in directum (by Prop. 14, B. 1), therefore complete the parallelogram DC.

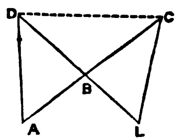
Because the parallelograms AD and GC are equal (by Hypoth.), AD is to DC, as GC to DC (by Prop. 14, B. 5); but AD is to DC, as AB to BC (by Prop. 1, B. 6); and GC is to DC as GB to BD, therefore AB is to BC, as GB to BD (by Prop. 18, B. 5).

PART 2.—The same construction remaining, AD is to DC, as AB to BC; and GC is to DC, as GB to BD; but AB is to BC, as GB to BD (by Hypoth.)

and therefore AD is to DC, as GC to DC (by Prop. 18, B 5), therefore AD is equal to GC (by Prop. 15, B. 5).

PROPOSITION XV. THEOREM.

Of equal triangles (ABD and CBL), having also an angle in the one equal to an angle in the other, the sides about the equal angles are reciprocally proportional (AB to BC as LB to BD). And if two triangles (ABD and CBL), have an angle of one equal to an angle of the other, and the sides about the equal angles reciprocally proportional, they will be equal to one another.

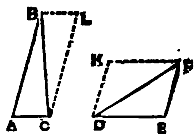


Part 1.—Let the two sides, AB and BC, about the equal angles be placed in directum, and because ABD and DBC are equal to two right angles (by Prop. 13, B. 1), and LBC is equal to ABD (by Hypoth.), DBC and LBC shall be equal to two right angles, and therefore DB and BL are in directum (by Prop., 14, B. 1); therefore draw DC.

Because the triangles ABD and LBC are equal (by Hypoth.), ABD is to DBC, as LBC to the same DBC (by Prop. 14, B. 5); but ABD is to DBC, as AB to BC (by Prop. 1, B. 6); and LBC is to DBC, as LB to BD (by Prop. 1, B. 6); therefore AB is to BC, as LB to BD (by Prop. 18, B. 5).

Part 2.—The same construction remaining, ABD is to DBC, as a AB to BC; and LBC is to DBC, as LB to BD; but AB is to BC, as LB to BD (by Hypoth.), and therefore ABD is to DBC, as LBC tJ DBC (by Prop. (18. B. 5); therefore ABD is equal to LBC (by Prop. 15, B. 5).

Cor. 1.—From this Proposition and the preceding, it appears that the triangles, or parallelograms, are equal, of which the bases and altitudes are reciprocally proportional.



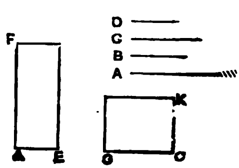
COR. 2.—If two triangles ABC and DFE have two sides reciprocally proportional to two, AB to EF as ED to AC, and the angles between those sides together equal to two right angles, the triangles shall be equal.

For let the parallelograms AL and DF be completed; and since the angles BAC and DEF are together equal to two right angles (by Hypoth.), and BAC and ACL are also equal to two right angles (by Prop. 29, B. 1), DEF and ACL will be equal; but BA is to EF as ED to AC (by Hypoth.), and CL is equal to AB (by Prop. 34, B. 1), and therefore CL is to EF as ED to AC, and the angles ACL and DEF are equal, therefore the parallelogram AL is equal to the parallelogram DF (by Prop. 14, B. 6), and therefore the triangles ABC and DEF, which are the halves of them (by Prop. 34, B. 1), are also equal.

COR. 3.— Similarly, if two equal triangles, ABC and DEF, have two angles, BAC and DEF, taken together, equal to two right angles, the sides about them will be reciprocally proportional.

PROPOSITION XVI. THEOREM.

If four right lines be proportional (A to B as C to D), the rectangle under the extremes (A and D) will be equal to the rectangle under the means (B and C). And if there be four right lines (A, B, C, and D) and the rectangle under the extremes be equal to the rectangle under the means, the four right lines shall be proportional.



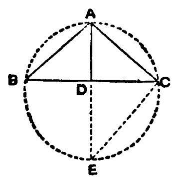
PART 1.—Draw AE and GC equal to D and C, and erect AF and CK perpendicular to them, equal to A and B, and complete the rectangles EF and GK. Therefore, since in the

parallelograms EF and GK, the angles A and C are equal, and the sides about them are reciprocally proportional (by Hypoth. and Const.), EF will be equal to GK (by Prop. 14, B. 6).

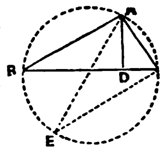
PART 2.—The same construction remaining, since the parallelograms EF and GK are equal (by Hypoth.), and the angles A and C are equal, AF will be to CK as GC to AE (by Prop. 14, B. 6), therefore A is to B as C to D (by Const.).

COR. 1.—If there be four right lines proportional, a parallelogram under the extremes, will be equal to an equiangular parallelogram under the means.

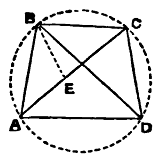
COR. 2.—In any triangle BAC, the rectangle under the two sides BA and AC, is equal to the square of the right line AD bisecting the angle contained by them, together with the rectangle under the segments, BD and DC, of the side opposite that angle.



Circumscribe a circle about the triangle BAC, whose circumference meets AD produced to E, and draw EC. Therefore, since the angle BAD is equal to the angle DAC (by Hypoth.), and the angles ABD and AEC, standing upon the same arch AC, are also equal (by Prop. 21, B. 3), the triangles BAD and EAC shall be equiangular; and therefore BA is to AD, as EA to AC (by Prop. 4, B. 6); therefore the rectangle under BA and AC is equal to the rectangle under EA and AD (by Prop. 16, B. 6), and therefore is equal to the rectangle under ED and DA, together with the square of DA (by Prop. 3, B. 2); but the rectangle under ED and DA, is equal to the rectangle under BD and DC (by Prop. 35, B. 3), therefore the rectangle under BA and AC, is equal to the rectangle under BD and DC, together with the square of AD.



COR. 3.—If from any angle A of a triangle BAC, a perpendicular AD be let fall on the opposite side, the rectangle under the sides, BA and AC, about that angle, will be equal to the rectangle under the perpendicular and the diameter of the circle about the circumscribed triangle. For let a circle be circumscribed about a triangle BAC, and draw the diameter AE, join EC. Therefore since the right angle ADB is equal to the angle ECA in a semicircle (by Prop. 31, B. 3), and the angles ABC and AEC standing on the same arch, are also equal (by Prop. 21, B. 3), the triangle BAD will be equiangular to the triangle AEC; and therefore BA is to AD, as AE tc AC (by Prop. 4, B. 6), therefore the rectangle under BA and AC is equal to the rectangle under AD and AE (by Prop. 16, B. 6).



COR. 4.—In any quadrilateral figure ABCD inscribed in a circle, the rectangle under the diagonals AC and BD will be equal to the rectangles under the opposite sides AB and CD, and under BC and AD.

Construct an angle ABE equal to the angle CBD, and in the triangles ABE and DBC, the angles ABE and DBC are equal, and also the angles BAE and BDC, standing upon the same arch, are equal (by Prop. 21, B. 3); therefore the angles ABE and DBC are equiangular, and therefore AB is to AE, as DB to DC (by Prop. 4, B. 6); therefore the rectangle under AB and DC is equal to the rectangle under AE and DB (by Prop. 16, B. 6); but since the angle ABE is equal to the angle CBD (by Const.), by adding the common angle EBD to both, the angles ABD and CBE shall be equal, and also the angles BDA and

BCE, standing upon the same arch, are equal (by Prop. 21, B. 3), therefore the triangles ADB and ECB are equiangular; and therefore AD is to DB, as EC to CB (by Prop. 4, B. 6), therefore the rectangle under AD and BC is equal to the rectangle under DB and EC (by Prop. 16, B. 6); but the rectangle under AB and DC is equal to the rectangle under DB and AE, and therefore the rectangles under AB and DC, and under AD and BC, will be equal to the rectangle under DB and AC (by Prop. 1, B. 2.)

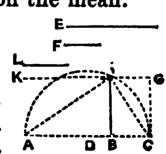
PROPOSITION XVII. THEOREM.

If three right lines be proportional (A to B as B to C), the rectangle under the extremes will be equal to the square which is constructed on the mean. And if the rectangle under the extremes be equal to the square constructed on the mean, the three right lines shall be proportional.

PART 1.—Assume D equal to B; therefore A will be to B as D to C (by Prop. 14, B. 5), and therefore the rectangle under A and C is equal to the rectangle under B and D (by Prop. 16, B. 6), and therefore equal to the square of B.

PART 2.—Assume D equal to B, and the rectangle under A and C will be equal to the rectangle under B and D; therefore A is to B as D to C (by Prop. 16, B. 6); and therefore A is to B as B to C (by Prop. 14, B. 5).

COR. 1.—If three right lines be proportional, any parallelogram under the extremes will be equal to an equiangular parallelogram constructed on the mean.



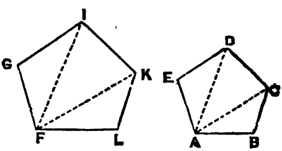
COR. 2.—A given right line can be divided into segments, AB and BC, the rectangle under which shall be equal to a given square, by dividing it so that the side of the square be a mean proportional between the seg-

ments of the given line (by Cor. 2, Prop. 13, B. 6); but it appears that this cannot be done, if the side of the square be greater than half the given line.

Cor. 3.—Hence it appears that the rectangle under two right lines is a mean proportional between their squares, for it is equal to the square of the mean proportional between them; but the squares of the proportional right lines are proportional.

PROPOSITION XVIII. PROBLEM.

Upon a given right line (AB) to describe a rectilineal figure similar to a given rectilineal figure, and similarly situated.

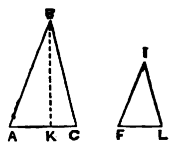


Draw FI and FK; with the right line AB, and at the points A and B, construct the angles BAC and ABC, equal to LFK and FLK; let the lines AC and BC meet in C, and the angle BCA shall be equal to the angle LKF (by Cor. 2, Prop. 32, B. 1); similarly, upon AC describe a triangle, equiangular to FKI, and so on. Therefore since the angle ABC is equal to the angle FLK (by Const.), and because the angle BCA is equal to the angle LKF, and ACD to FKI (by Const.), BCA is also equal to LKI; and so on, it can be shown that each angle in the figure AEDCB, is equal to each angle in FGIKL; therefore AEDCB is equiangular to the given figure FGIKL; but the triangle ABC, is equiangular to the triangle FLK (by Const.) and therefore AB is to BC, as FL to LK (by Prop. 4, B. 6); and also BC to CA, as LK to KF (by Prop. 4, B. 6); but ACD is equiangular to FKI, and therefore CA is to CD, as FK to KI (by last ref.), and therefore *ex æquali* (by Prop. 34, B. 5), BC to CD, as LK to KI; and it can be similarly shown that the sides about the

remaining angles are proportional, and since the figures AEDCB and FGIKL are also equiangular, they shall be similar to each other.

PROPOSITION XIX. THEOREM.

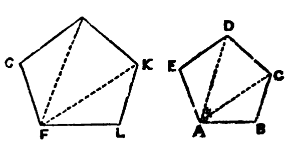
Similar triangles (ABC, FIL) are to one another in the duplicate ratio of their homologous sides.



Assume a third proportional KC to the homologous sides, AC and FL, and draw BK. Therefore, since AC is to CB, as FL to LI (by Hypoth.), by alternation, AC will be to FL, as CB to LI (by Prop. 33, B. 5); but AC is to FL, as FL to CK (by Const.); therefore CB is to LI, as FL to CK (by Prop. 18, B. 5); but the angle C is equal to the angle L (by Hypoth.), and therefore the triangle KBC is equal to the triangle FIL (by Prop. 15, B. 6); therefore ABC has to both, the same ratio (by Prop. 14, B. 5); but ABC is to KBC, as AC to KC (by Prop. 1, B. 6), therefore ABC is to FIL, as AC to KC, or in a duplicate ratio of AC to FL (by Def. 10, B. 5).

PROPOSITION XX. THEOREM.

Similar Polygons (FGIKL and AEDCB) may be divided into similar triangles, equal in number, and homologous to all. And the polygons are to one another in the duplicate ratio of their homologous sides.



PART 1.—Because in the triangles FGI and AED, the angles G and E are equal, and the sides about them proportional (by Hypoth. and Def. 1, B. 6), FGI and AED shall be similar (by Prop. 6, B. 6).

therefore, since the angles GIF and EDA are equal, but GIK and EDC are equal to the whole (by Hypoth. and Def. 1, B. 6), the remainders FIK and ADC will be equal; and since FI is to IG, as AD to DE, and IG to IK, as DE to DC (by Hypoth. and Def. 1, B. 6), *ex æquali*, FI will be to IK, as AD to DC (by Prop. 33, B. 5); and therefore, because the angles contained by them are equal, the triangle FIK shall be similar to the triangle ADC (by Prop. 6, B. 6), and that the remaining triangles are also equal, can be similarly proved.

PART 2.—Because the triangle FGI is similar to the triangle AED, FGI will be to AED in the duplicate ratio of the sides FI to AD (by Prop. 19, B. 6); also FIK is similarly to ADC in the duplicate ratio of FI to AD; therefore FGI is to AED, as FIK is to ADC (by Prop. 18, B. 5); and thus it can be also shown that FIK is to ADC, as FKL to ACB; therefore as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents (by Prop. 22, B. 5), or the polygon FGIKL, to the polygon AEDCB.

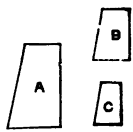
PART 3.—Because the polygon FGIKL is to the polygon AEDCB, as the triangle FGI is to the triangle AED, and FGI is to AED, in the duplicate ratio of the side FG to the side AE (by Prop. 19, B. 5), FGIKL will be also to AEDCB, in the duplicate ratio of FG to AE (by Prop. 18, B. 5).

COR. 1.—Hence if three right lines be proportional, a rectilineal figure upon the first, will be to a similar, and similarly situated figure upon the second, as the first to the third.

COR. 2.—Hence a rectilineal figure can be described, which is to a given one in any given ratio; for find a mean proportional between the side of the figure and the right line which is to it, in the given ratio, and upon it describe a figure similar to the given one, and similarly situated.

PROPOSITION XXI. THEOREM.

Rectilineal figures (A and B) which are similar to the same, (C) are also similar to one another.

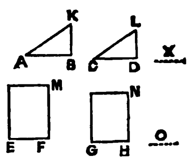


For since the rectilineal figure A is similar to the rectilineal figure C, it is equiangular to it, and has the sides about the equal angles proportional (by Def. 1, B. 6), and since the rectilineal figure B, is similar to C, it is equiangular to it, and has the sides about the equal angles proportional (by Def. 1, B. 6); therefore A and B are also equiangular to each other (by Ax. 1, B. 1), and have the sides about the equal angles proportional (by Prop. 18, B. 5), therefore A and B are similar.

PROPOSITION XXII. THEOREM.

If four right lines be proportional (AB to CD, as EF to GH), the similar rectilineal figures, and similarly described on them shall be also proportional.

And if similar rectilineal figures, similarly described on four right lines, are proportional, those right lines shall be proportional.



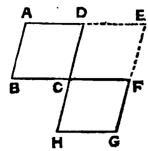
PART 1.—Assume a third proportional X to AB and CD, and a third proportional O, to EF and GH; because AB is to CD, as EF to GH (by Hypoth.), CD will be to X, as GH to O (by Prop. 18, B. 5); and therefore *ex æquali*, AB will be to X, as EF to O (by Prop. 33, B. 5); but AKB is to CLD, as AB to X (by Cor. 1, Prop. 20, B. 6); and EM to GN, as EF to O; therefore, AKB is to CLD, as EM to GN.

PART 2.—Letting the same construction remain; since AKB is to CLD, as EM to GN (by Hypoth.),

AB will be to X, as EF to O (by Prop. 18, B. 5); therefore AB is to CD as EF to GH (by Prop. 39, B. 5).

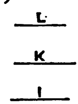
PROPOSITION XXIII. THEOREM.

Equiangular parallelograms (BD and CG) have to each other the ratio compounded of the ratios of their sides.



Place the two sides BC and CF about the equal angles in directum; and since the angles BCD and DCF are equal to two right angles (by Prop. 13, B. 1), but HCF is equal to BCD (by Hypoth.), DCF and FCH will be also equal to two right angles, and therefore DC and CH are in directum (by Prop. 14, B. 1), therefore complete the parallelogram CE.

Therefore, because BD is to CE, as BC to CF (by Prop. 1, B. 6), and CE to CG, as DC to CH (by same), DB has to CG a ratio compounded of the ratios of BC to CF, and DC to CH (by Def. 12, B. 5).



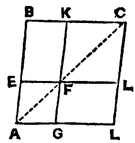
Cor. 1.—Two right lines can be found, which are to one another in the ratio of the parallelograms BD and CG; by assuming any right line I, and by finding a fourth proportional K, to BC, CF, and I; and a fourth proportional L, to DC, CH, and K, and BD will be to CE, as I to K; and CE to CG, as K to L; and therefore *ex æquali*, BD will be to CG, as I to L (by Prop. 33, B. 5).

Cor. 2.—Triangles which have one angle of the one equal to one angle of the other, are to each other in a ratio compounded of the ratios of the sides about the equal angles.

Cor. 3.—Any parallelograms or triangles are to one another in a ratio compounded of the ratios of their bases and altitudes; for they are equal to rectangles, or to right-angled triangles, upon equal bases, and of the same altitude.

PROPOSITION XXIV. THEOREM.

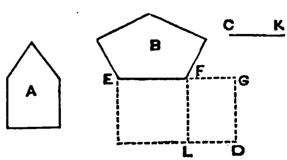
In every parallelogram (AC) the parallelograms which are about the diagonal (AF and FC) are both similar to the whole and to each other.



For since the parallelograms AC and AF have a common angle A, they are equiangular (by Cor. 2, Prop. 34, B. 1); but on account of the parallels EF and BC, the triangles AEF and ABC are similar (by Cor. 1, Prop. 4, B. 6), and therefore AE is to EF, as AB to BC (by Def. 1, B. 6); but the remaining sides are equal to AE, EF, AB, and BC (by Prop. 34, B. 1); therefore the parallelograms AF and AC have the sides about the equal angles proportional, and therefore are similar (by Def. 1, B. 6). It can be similarly demonstrated that AC and FC are similar. Therefore, since each of the figures AF and FC is similar to the same AC, they will be similar to each other (by Prop. 21, B. 6).

PROPOSITION XXV. PROBLEM.

To construct a rectilineal figure, equal to a given one (A), and similar to another (B).



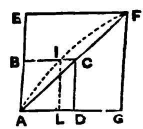
Upon any side EF of the given figure B construct a rectangle EL, equal to B (by Cor. 1, Prop. 45, B. 1), and on its side FL construct a rectangle FD, equal to the given one A (by Cor. 1, Prop. 45, B. 1); find a mean proportional CK between the other sides EF and FG (by Prop. 13, B. 6), then the figure described upon it, similar to the given one B, and similarly situated, will be equal to the given figure A. For the rectangle EL is to the rectangle FD, as EF to FG (by Prop. 1, B. 6);

or in the duplicate ratio of EF to CK (by Const.); and therefore, as the rectilineal figure B to the similar and similarly described one upon CK (by Prop. 20, B. 6); but EL is equal to B (by Const.), therefore the rectilineal figure described upon CK, similar to B, and similarly situated, will be equal to FD, and therefore to the given figure A (by Prop. 23, B. 5).

COR.—Hence appears a method of constructing a rectilineal figure, similar to a given one, and equal to the sum or difference of two others; if we first construct a parallelogram equal to that sum or difference (by Cors. 2 and 3 Prop. 45, B. 1).

PROPOSITION XXVI. THEOREM.

If similar and similarly situated parallelograms (AC and AF), have a common angle, they are about the same diagonal.

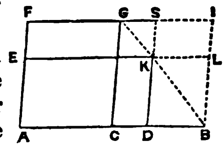


For if not, but if possible, let AIF be the diagonal of the parallelogram AF, and through I draw IL parallel to AE.

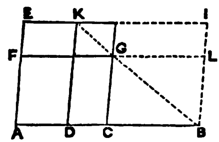
Therefore since the parallelograms AI and AF, are about the same diagonal AIF, and have the angle A common to both, AI and AF shall be similar (by Prop. 24, B. 6), therefore BA is to AL, as EA to AG; but BA is to AD, as EA to AG (by Hypoth.); therefore BA is to AL, as BA to AD (by Prop. 18, B. 5); and therefore AL is equal to AD (by Prop. 15, B. 5) which is absurd. Therefore AIF is not the diagonal of AF, and it can be similarly shown that no other line is a diagonal except ACF.

PROPOSITION XXVII. THEOREM.

If any right line (AB) be cut equally (in C) and unequally (in D), the parallelogram (FC) which is applied to the half deficient by a figure (GB) similar to itself, will be greater than the parallelogram (ED), applied to either of the other parts, whose deficiency (KB) is similar to the first (GB).



First, let AD be the greater segment of the right line AB, and complete the parallelogram KI, and draw GB. Therefore since GB and KB are similar (by Hypoth.), GB will be the diagonal of both (by Prop. 26, B. 6); therefore CK is equal to KI (by Prop. 43, B 1), and therefore if DL be added to both, CL will be equal to DI; but CL and CE are equal (by Hypoth.), and therefore CE and DI are equal; add CK to both, and DE will be equal to the gnomon CLS, and therefore is less than the whole parallelogram CI; that is, less than the parallelogram FC, which is equal to CI (by Hypoth. and Prop. 36, B. 1).



Now let AD be the less segment, complete the parallelogram GI, and draw KB; and since the parallelograms KB and GB are similar (by Hypoth.); they shall be about the same diagonal (by Prop. 26, B. 6); and therefore the parallelograms DG and GI are equal (by Prop. 43, B. 1); but FG and GL are equal (by Hypoth. and Prop. 34, B. 1), therefore the parallelograms EG and GI are equal (by Prop. 36, B. 1); but EG is greater than FK, therefore GI will be also greater than FK; and therefore DG, which is equal to GI, is greater than FK, therefore add FD to both, and the whole FC will be greater than the whole ED.

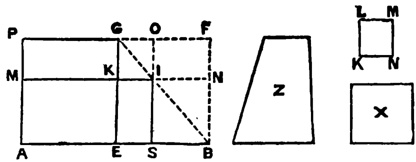
Schol.—It appears that the parallelogram KG is the excess of the parallelogram FC above ED, and it is diminished, DC being diminished.

PROPOSITION XXVIII. PROBLEM.

To a given right line (AB), to apply a parallelogram equal to a given rectilineal figure (Z) and deficient by a figure similar to a

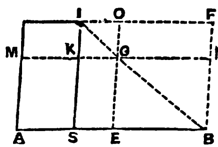
given parallelogram (X). But the rectilineal figure must not be greater than the parallelogram applied to half the given line, whose defect is similar to the given figure (X).

Bisect AB in E, and upon AE describe a parallelogram AG, similar to the given figure X, and complete



APFB. AG will be either equal to, or greater than the given rectilineal figure Z, (by the conditions of the problem). If it be equal the problem is done. If it be greater, construct a parallelogram equal to its excess above Z, similar to X (by Cor. Prop. 25, B. 6), and let it be KLMN; and since this is less than the parallelogram AG, it will be less than EF, which is equal to AG (by Const. and Prop. 36, B 1); but it is similar to it, and therefore its sides KL and LM, are less than the homologous sides EG and GF, of the parallelogram EF; therefore take away from these GK and GO equal to KL and LM, and complete the parallelogram KGOI, and this will be similar to the parallelogram EF, since each is similar to the same X, (by Const.), and it is also similarly situated, therefore KGOI and EF are about the same diagonal (by Prop. 26, B. 6); draw their diagonal GIB, and produce OI to S, and KI to M and N; because the parallelogram EF, is equal to KLMN together with Z (by Const.), but KO is equal to KLMN, the gnomon ENO shall be equal to Z; but EI and IF are equal (by Prop. 43, B. 1), therefore if SN be added to both, EN and SF are equal; but because AE and EB are equal, EN is equal to ME (by Prop. 36, B. 1), therefore ME and SF are equal, and therefore if EI be added to both,

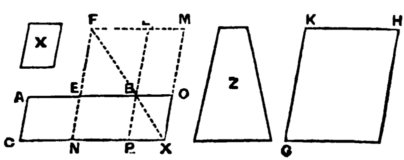
MS will be equal to the gnomon ENO, but the gnomon ENO is equal to the given rectilineal figure Z, therefore MS is equal to the given figure Z, and its defect SN, is also similar to the given figure X (by Const. and Prop. 21, B. 6), because the parallelogram EF is similar (by Prop. 24, B. 6), therefore what was proposed is done.



SCHOL.—By this method a parallelogram shall be constructed upon the greater segment of the given line AB; but it can be constructed upon the less, if the lines GO and GK be taken away from the right line EG produced, and from GM; and having described the figure as in the proposition, AI will be the parallelogram required, as is easily demonstrated; for KO is the excess of AG above AI (by Schol. Prop. 27, B. 6), it is also its excess above the given figure Z, therefore AI and Z are equal.

PROPOSITION XXIX. PROBLEM.

To a given right line (AB) to apply a parallelogram, equal to a given rectilineal figure (Z), and exceeding by a figure (BX), similar to a given parallelogram (X).

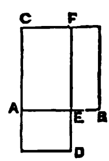


Bisect AB in E, and upon EB construct a parallelogram, similar to the given parallelogram X; and construct a parallelogram GH similar to the parallelogram EL, but equal to EL together with the given figure Z (by Cor. Prop. 25, B. 6); and since GH is greater than EL, its sides, GK and KH, will be

greater than the homologous sides of EL, which are FE and FL; therefore, on those sides produced take FN and FM, equal to GK and KH, and complete the parallelogram NM; this will be similar to the parallelogram GH, and therefore similar to EL, but it is similarly situated, and they are therefore about the same diagonal; draw their diagonal FBX, and through A draw AC parallel to EN, meeting PN produced; and since NM and GH are equal, and GH is equal to Z and EL taken together (by Const.), NM is also equal to Z and EL together; take away EL from both, and the gnomon NOL will be equal to Z; but because AE and EB are equal, the parallelograms AN and EP are equal (by Prop. 36, B. 1); and EP and BM are also equal (by Prop. 43, B. 1), therefore AN is equal to BM; let the common part NO be added, and AX will be equal to the gnomon NOL, and therefore AX is equal to the given rectilineal figure Z; but its excess PO is similar to the parallelogram EL, and therefore to the given figure X.

PROPOSITION XXX. PROBLEM.

To cut a given finite right line (AB) in extreme and mean ratio.



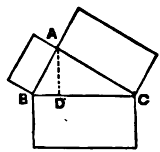
Upon AB describe a square BC (by Prop. 46, B. 1), and to AC apply a parallelogram equal to BC, exceeding by a figure AD similar to it (by Prop. 29, B. 6); and since AD is similar to BC, it will be a square; and because BC and CD are equal, by taking away the common rectangle CE, BF will be equal to AD; but it is also equiangular to it, therefore EF is to ED, as EA to EB (by Prop. 14, B. 6); but EF and ED are equal to AB and AE, therefore AB is to AE, as AE to EB.

Otherwise,

Cut AB in E, so that the rectangle under AB and EB be equal to the square of AE, (by Prop. 11, B. 2), and as AB is to AE, so will AE be to EB (by Prop. 17, B. 6), therefore AB is cut in extreme and mean ratio.

PROPOSITION XXXI. THEOREM.

If any similar rectilineal figures be similarly described on the sides of a right angled triangle (ABC) that figure which is described upon (BC) opposite the right angle, will be equal to the two remaining figures taken together.

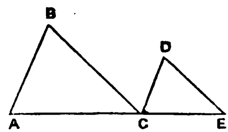


From the right angle draw a perpendicular AD to the opposite side; and BC will be to CA, as CA to CD (by Cor. Prop. 8, B. 6); therefore the figure upon BC is to the similar figure upon CA, as BC to CD; (by Cor. 1, Prop. 20, B. 6); similarly, the figure upon BC is to the similar figure upon BA, as BC to BD; therefore the figure upon BC, is to the two figures upon AC and AB together, as BC, to BD and DC taken together; but BC is equal to BD and DC; and therefore the figure upon BC, is equal to the similar figures upon BA and AC taken together (by Cor. Prop. 24, B. 5).

Cor.—Hence any number of similar rectilineal figures being given, a rectilineal figure can be found, by Cor. 2. Prop. 47, B. 1, which is equal to the sum of the whole; and by Cor. 3, Prop. 47, B. 1, another can be found equal to the difference of two given similar figures. But if the given figures be not similar, construct figures similar to one of them, and equal to each of the others, and then a figure can be found equal to their sum or difference.

PROPOSITION XXXII. THEOREM.

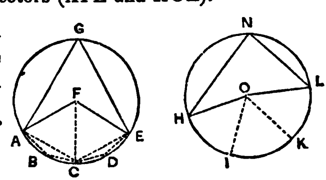
If two triangles (ABC, CDE) have two sides of the two proportional (AB to BC as CD to DE), and be so placed at an angle, that the homologous sides be parallel, and that the sides not homologous (CB and CD) contain the angle at which they are placed, the remaining sides (AC and CE) shall be in directum.



For since AB and CD are parallel, the alternate angles B and BCD are equal (by Prop. 29, B. 1); and similarly, because CB and ED are parallel, the angles D and BCD are equal (by same); therefore B is equal to D; and because the sides about these angles are proportional (by Hypoth.), the triangles ABC and CDE shall be equiangular (by Prop. 6, B. 6), therefore the angles ACB and CED are equal; but BCD is equal to CDE; and therefore if DCE be added to both, ACD and DCE together will be equal to CED, EDC, and DCE; therefore ACD and DCE are equal to two right angles (by Prop. 32, B. 1), and therefore AC and CE are in directum (by Prop. 14, B. 1).

PROPOSITION XXXIII. THEOREM.

In equal circles, angles whether at the centre (AFE and HOL), or at the circumference (AGE and HNL), have the same ratio, that the arches on which they stand have to each other. So also have the sectors (AFE and HOL).



First let the given angles AFE and HOL be at the centres, and divide the arch ACE into any number of equal parts, AC and

CE, and assume HI, IK and KL equal to AC; and draw FC, OI and OK. Therefore, because the arches AC, CE, HI, IK and KL are equal (by Constr.), the angles AFC, CFE, HOI, IOK and KOL are also equal (by Prop. 27. B. 3.); therefore such a submultiple as the arch AC is of ACE, such will the angle AFC be of AFE; and as often as the arch AC is contained in HIKL, so often is the angle AFC contained in HOL; therefore ACE is to HILK, as AFE to HOL (by Def. 5. B. 5).

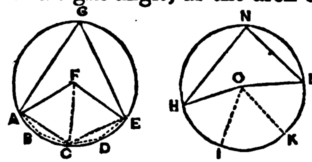
If the given angles AGE and HNL be at the circumference, it can be similarly demonstrated that AGE is to HNL, as the arch ACE is to the arch HIKL.

But the sector AFE is also to the sector HOL, as the arch ACE to the arch HIKL.

For letting the preceding construction remain, take any points B and D in AC and CE, and draw BA, BC, DC and DE; and since, if the equal arches AC and CE be taken away from the circle AGEC, the remainders AGC and CGE are equal, the angles ABC and CDE will be equal (by Schol. Prop. 20, B. 3), and therefore the segments ABC and CDE are similar (by Def. 10, B. 3); but because the arches ABC and CDE are equal (by Const.), AC, and CE subtending them are also equal (by Schol. Prop. 20, B. 3), therefore the segments ABC and CDE are equal (by Prop. 24, B. 3), but the triangles AFC and CFE are equal, because the angles at F, and the sides about them are equal: therefore the whole sector AFC is equal to the whole CFE; and it can be similarly shown that these others HOI, IOK and KOL are equal; therefore such a submultiple as the arch AC is of ACE, such will be the sector AFC of AFE; but so often as the arch AC is contained in HIKL, so often is the sector AFC contained in HOL; therefore ACE is to HIKL, as the sector AFE to the sector HOL (by Def. 5, B. 5).

COR. 1.—An angle AFE at the centre, is to four right angles, as the arch on which it stands, is to the whole circumference.

For the angle AFE is to a right angle, as the arch on which it stands, to the fourth part of the circumference (by Prop. 33, B. 6); and therefore to four right angles, as the arch to the whole circumference (by Prop. 30, B. 5).



COR. 2.—Of unequal circles, the arches which subtend the equal angles, are similar.

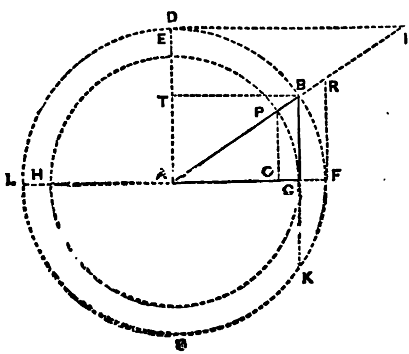
For it appears that they have the same ratio to the whole circumference (by preceding Cor.)

COR. 3.—Hence it appears that arches, by which similar segments are contained, are similar.

END OF SIXTH BOOK.

TRIGONOMETRY.

TRIGONOMETRY is of two kinds, Plane and Spherical, it treats of the measurement of heights, distances, &c., by means of triangles. Plane Trigonometry, on which we shall comment, is divided into two parts, scilicet, right angled, and oblique, since all triangles must be either rectangular or otherwise. A triangle consists of six parts; id est, three angles and three sides. If any three of these parts be given, provided one be a side, the remaining three can be found; by means of the triangle undergoing the following change; thus if the vertex of either of the acute angles of a right-angled triangle be taken as a centre, and any of the sides as radius, a circle being described, the sides of the triangle receive the following names; if the radius be the longest side: *sine, cosine,* and *radius;* and the following, if the radius be either of the other sides: *tangent, secant,* and *radius.* It must be remembered that they retain



their names with regard to the angle which has been taken as a centre, therefore let ABC be a triangle, and taking AB as radius, and describing the circle BDLGF,

we have BC, the sine of the angle BAC, and AC the cosine of the same angle, but it must be also understood that the arc intercepted between the legs of an angle is the measure of that angle, so that if a circle be divided into 360 degrees, the arc intercepted between the legs of a right angle will be 90 degrees, and so every angle is said to be an angle of as many degrees as are contained in an arc intercepted between its legs. Again, if we take a minor side of the triangle, as radius, and the same angle as centre, and describe the circle CEH, we have the tangent BC of the angle BAC, and AB the secant of the same angle BAC. The circle is usually divided in ENGLAND into 360 degrees, but in FRANCE into 400 degrees; this mode of division has an advantage over that of England, since when degrees, minutes, seconds, and thirds are expressed, which are thus written, $20^{g}\ 8'\ 7''\ 6'''$, we could also express them in the decimal notation by the very same numbers thus 20·876.

The complement of an angle is what it wants of a right angle, and its sine, tangent, &c., are called the co-sine or sine of the complement, &c. The supplement is what an angle wants of two right angles or a semicircle—and the sine, &c. of the supplement is the same as the sine, &c. of that arc or angle.

Now taking the angle BAF of the triangle BAC, or its measure which is either the arc BF or PC, we shall proceed to explain all the lines which can be drawn to the circle DLGF with regard to that angle, that is, all the lines used in Trigonometry, as a Cord, derived from *Corda*, the string of a bow; sine, from *sinus*, the bosom; versed sine, from *sagitta*, an arrow; tangent, being the line touching the circle; secant, the line cutting the circle, and the complements of these, which are called co-sine, co-cord, &c. being merely the sine, cord, &c. of the complement of the angle in question.

Thus the sine BC of the angle BAC, or the arc BF, is a line perpendicular to the radius, and meeting an

extremity of the arc BF. The sine of BD, or the co-sine of BF, is a line BT, drawn from the same extremity of the arc, parallel and equal to AC. The tangent is a line RF drawn perpendicularly from the extremity of the radius, meeting the other leg of the angle in R, which is called the secant or cutting line, since it cuts the circle in B when produced. The line DI is the tangent of the angle EAB or the arc DB, but it is also the co-tangent of the angle BAF. The line AI meeting DI in I is called the secant of EAB, or the co-secant of BAF. The line CF, between the sign and tangent, is called the versed line. It must be now remembered that the sine, tangent, &c. of any arc is also the sine, &c. of its supplement, and that the sine of an arc is half the cord of double that arc, which is evident since BFK is double of BF, and BC is half of BK.

Next, in treating analytically of Trigonometry, it will be requisite to alternately consider the above lines as plus or minus, according as they become nothing or infinite, when removed from the commencement of the first quadrant round the circle.

In the first quadrant the lines are all considered as +, then if we take the sine and move it on through the semicircle in which it commences we find it will not become nothing until at the end of the semicircle, therefore the sine is + in the two first quadrants, and − in the two second; and likewise + in the fifth quadrant, and similarly any number of times round the circle. Again if we take the co-sine, which is the radius at the commencement of the first quadrant, and move on towards the centre, we find it nothing on arriving at that point, therefore it becomes − in the second quadrant, and also in the third, because it does not again become nothing till it returns to the centre, therefore it will be + in the fourth quadrant, and similarly any number of times round the circle. Next let us take the tangent, which is nothing at the beginning of the first quadrant, but increases until the arc becomes a quadrant, when it

is evident that it changes its sign, since it becomes infinite, because it will be parallel to the line which limits it (the secant) and therefore the tangent will be − in the second quadrant, and similarly in each alternate one. The secant is radius at the commencement of the first quadrant, it increases by limiting the tangent, till it becomes infinite, when the angle becomes right, and therefore changes its sine to − in the second quadrant, and retains the same in the third, because it neither becomes infinite nor nothing at the end of the second; but since it becomes infinite at the end of the third, it will also change its sign; therefore the fourth quadrant will be +, and so on in the same order, any number of times round the circle. The co-tangent undergoes the same change as the tangent, and the co-secant the same with the sine, therefore as often as those lines become nothing or infinite, they also change their sines. Lastly, the versed sine being either the diameter or radius, it always retains the sine plus.

On the next page we give, at one view, the values at the beginning and ending of each quadrant—but it must be particularly observed, that the lines never change their signs, when moved round the circle, unless they become nothing or infinite.

THEREFORE WE HAVE THE FOLLOWING VALUES AT THE BEGINNING AND END OF EACH QUADRANT:—

Degrees of the Circle.	Sin.	Co-Sin.	Tan.	Co-Tan.	Sec.	Co-Sec.	Ver-Sin.
0°	nothing	+ rad	nothing	+ inf.	+ rad	+ inf.	nothing
90°	+ rad	nothing	+ inf.	nothing	+ inf.	+ rad	+ rad
180°	nothing	− rad	nothing	− inf.	− rad	− inf.	+ diam.
270°	− rad	nothing	− inf.	nothing	− inf.	+ rad	+ rad
360°	nothing	+ rad	nothing	− inf.	+ rad	+ inf.	nothing

We shall now disregard the circle CEH, and proceed to analyze the triangles formed by the lines drawn with regard to the circle FDLG.

In the figure we have three right angled, similar triangles.

The first is formed by the radius, sine and co-sine.

The second . . by the secant, tangent, and radius.

The third . . by the co-secant, radius and co-tangent.

In the first triangle we have (by Euclid, Prop. 47, B. 1), $\text{sin.}^2 + \text{co-sin.}^2 = \text{rad.}^2$, or by making radius = 1 we have $\text{sin.}^2 + \text{co-sin.}^2 = 1$.

In the second we have, $\text{sec.}^2 = \text{rad.}^2 + \text{tan.}^2$ or $1 + \text{tan.}^2$.

In the third $\text{co-sec.}^2 = \text{co-tan.} + 1$.

Again (by Euclid, Prop. 4, B. 6), we have in the first and second triangles,

$$\text{sin.} : \text{co-sin.} :: \text{tan.} : 1.$$

Then $\dfrac{\text{sin.}}{\text{co-sin.}} = \text{tan.}$

And $\text{sin.} : 1 :: \text{tan.} : \text{sec.}$

Then $\text{sin.} = \dfrac{\text{tan.}}{\text{sec.}}$

Again $\text{cos.} : 1 :: 1 : \text{sec.}$

Then $\text{cos.} = \dfrac{1}{\text{sec.}}$

Then in the second and third triangles,

$$\text{tan.} : 1 :: 1 : \text{cot.}$$

Therefore $\text{tan.} = \dfrac{1}{\text{cot.}}$

And again $\text{cot.} : 1 :: \text{cos.} : \text{sin}$

Then $\text{cot.} = \dfrac{\text{cos.}}{\text{sin.}}$

From this it is easy to trace the relation between those lines, *ex. gr.* tan. $= \frac{\text{sin}}{\text{cos.}} = \frac{1}{\text{cot.}}$ &c.

The limits of this introduction will not admit of further investigation, but we trust it may simplify other works to the inquiring Student. We shall therefore proceed to give some examples in its application.

In the first place let the triangle be ABC as given in the first figure, then (by Euc. Prop. 4, B. 6,) we have AB : BC : : 1 : sin A, and AB : AC : : 1 : cos A, therefore the hypothenuse is to either leg, as the radius to the sine of the angle opposite the same leg, or to the cos of the adjacent angle. Let the three sides of the triangle be given to find the angles, thus AC = 172·1 unites, AB = 250, BC = 181·4.

Then by Logarithms we have,

As AB	250	2·39794
is to AC	172·1	2·23575
So is Radius	90°	10·00000
To the Cos. of A 46° 30′		9·83781

Since the angle at C is right, it is evident that 90 − A = B (by Euclid, Prop. 32, B. 1). The angle A could be similarly found bv AB : BC : : 1 : sin A as above.

Again if we make AC radius, and describe the circle CEH, the side CB becomes the tangent, and AB the secant of A, therefore we have AC : CB : : 1 : **tan. A,** by which we can also find the angle A, thus

As AC 172·1	2·23575
Is to CB 181·4	2·25850
So is radius 90°	10·00000
To Tan of A 46° 30′	10·02275

And since CA : AB : : 1 : sec. A, the angle A can be similarly found—therefore whatsoever side we make radius, the angles can be found. In the next place if there be an angle and a side given to find the other sides.

Wherefore suppose we have the angle A = 46° 30′, and the side AB = 250.

As Radius . . 90°	. . . .	10·00000
Is to the Sin 46° 30′	. . .	9·86056
So is AB . . 250	. . .	2·39794
To BC . . . 181·4	. . .	2·25850

It could be similarly found by making any other side radius.

If we wish to find the height of any perpendicular object, it appears that if we measure the distance from any point, in a horizontal plane, to its base, and take the angle of elevation at that point, this will involve the first case of right-angled trigonometry.

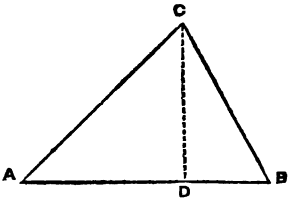
Therefore if we require the height of a spire, whose distance from any point, in a horizontal plane, to its base, is 85 feet, and the angle to its top from that point 53 degrees, we thus find its height. Let AO, in the first figure be 85 feet, the angle at A = 53 degrees, and PO the height required, then

As the Cos A 53°	. . . .	9·77946
Is to Sin A 53°	. . . . .	9·90235
So is AO 85 feet	. . . . .	1·92943
To PO	112·8 feet . .	2·05231
Observer's eye	5·0 feet.	
Height of object	117·8	

OBLIQUE ANGLED TRIANGLES

Require to be considered thus: let ABC be any triangle, and CD its perpendicular.

Therefore since in the triangles ACD, BCD, the perpendicular CD is equal to AC sin A (by making AC radius) and also equal to BC sin B (by making BC radius) AC will be to BC as sin B to sin A, that is, in every triangle the sides are proportional to the sines of their opposite angles, so that BC : CD :: sin A to sin B.



Now in the triangle ACB let us have two sides (AC = 240, BC = 200), and the angle opposite to either of them (A = 46° 30′) to find the remaining side and angles. Therefore to find the angle C, we have

As BC 200	2·30103
Is to CA 240	2·38021
So is sin A 46° 30′	9·86056
To the Sin of B 60° 31′	9·93974

It is evident that 180° – (A + B) = the angle C = 72° 59′. Next to find the side AB, we have

As sin A 46° 30′	9·86056
Is to sin C 72° 59′	9·98056
So is BC 200	2·30103
To AB 263·7	2·42103

It is now evident that we have the six parts of which every triangle is comprised.

The sides can be similarly found if two angles and

one side be given, bearing in mind that the third term of the proportion must be of the same name as the fourth, or required term.

Again when two sides and their contained angle are given, it will be necessary to consider it thus: let the triangle be ABC; and AB = 240, AC = 263·7, and the angle at A = 46° 30′, then the formula will be 240 + 263·7 : 263·7 − 240 : : tan. ½ B + C : tan. ½ B − C.

Therefore as 46° 30′ subtracted from 180° gives the sum of the other two angles, we have

As the Sum of the sides	= 503·7	2·70217
Is to their diff.	= 23·7	1·37475
So is the tan. of the half sum of the angles C and B	= 66° 45′	10·36690
To the Tan of half the difference	6° 15′	9·03948

Therefore this half difference = 6° 15′ + 66° 45′ = 73° 00′ the greater angle C, and 66° 45′ − 6° 15′ = 60° 30′ the less angle B.

Now having the sides about one angle and also the angles opposite those sides, it is evident we can find the other side by last case.

The next case is when three sides are given to find the angles.

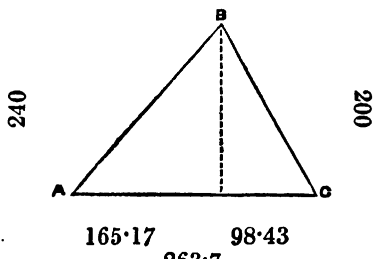
Then in the triangle ABC following, we have AB = 240, BC = 200, AC = 263·7

First let fall a perpendicular from the greater angle, then

As the side opposite to that angle B	263·7	2·42111
Is to the sum of the other side	440	2·64345
So is the difference of those sides	40	1·60206
To the difference of the segments of the base	66·74	1·82440

Then take half this difference = 33·37 and add it to half the greater side (131·8 + 33·37) = 165·17, the greater segment of the base; next (131·8 — 33·37) = 98·43 the less segment.

Now it is evident that we have the triangle divided into two right angled triangles, having two sides given in each and the angle opposite one of them—therefore it is manifest that this case can now be performed by the first case of right angled trigonometry, thus



Making AB radius
As AB . . 240 . . . .
Is to the greater Seg. 165·17 . . .
So is Radius . . .
To the Cos. of A.

The angle C can be similarly found; therefore 180° — (A + C) = B. We will leave this question for the pupil to fill up: its proof is evident, since all the angles of the triangle must be equal to two right angles, or 180°.

Example 1.

At 170 feet from the bottom of a tower, the angle of its elevation is found to be 52° 30′; required the altitude.

Ans...221·55 feet.

Example 2.

If the angle of depression of a ship's bottom, taken from the top of a tower 143 feet high, be 35°; what would be the distance of the ship from the bottom of the tower?

Ans...204·22 feet.

Example 3.

Being on the bank of the river Wye, near Monmouth, and wanting to know the distance to a house on the other side, we measured 200 yards in a straight line by the side of the river, and then took the contained angles between this line and the house—that is, the angle at one point being 68° 2′, and at the other, 73° 15′; required the distance from each point.

Ans...306·19, and 296·54.

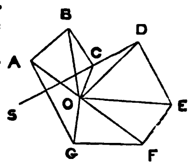
Example 4.

If 41 feet were taken from Salisbury Cathedral the remainder would be the height of St. Paul's, and if 2 feet were taken from the same spire the residue would be equal to twice the height of the Monument; but if it be also found that if 50 feet were taken from Salisbury spire the remainder would be equal to 154 feet, together with the height of the Monument; we demand the actual height of each, and what length of wire would reach to the top of St. Paul's, from a point 365 feet distant horizontally from the base of that Cathedral.

Ans...St. Paul's . . . . 365 feet.
Salisbury Spire . . 406 „
Monument . . . . 202 „
Hypothenuse required 516·18

APPENDIX.

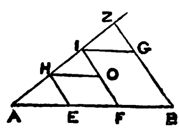
I. Some polygons have what is called a re-entrant angle, as BCD in the opposite figure. All the internal angles will still be $= 2n - 4$ (n being the number of sides), provided we consider the re-entrant angle as greater than 2 right angles, and in fact, equal to the excess of 4 right right angles above the angle BCD.



In this case, however, the external angles will not, as in an ordinary polygon, $=$ 4 right angles; but, considering BCS as the angle external to tho re-entrant angle, all the other external angles, minus BCS, will be equal to 4 right angles: for, we can no longer say that the internal angle $+$ the external adjacent, is equal to 2 right angles; for, in the case of the re-entrant angle, the internal minus the external adjacent is equal to 2 right angles.

In fact, the external angle corresponding to the re-entrant internal, is to be reckoned as negative, in taking the sum of the external angles.

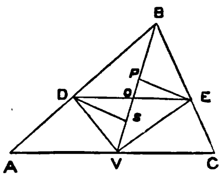
II. Clavius has given the following method of dividing a right line, AB, into any number of parts. (Suppose three parts.) Draw AZ at any angle; take any point on it, H, and make HI and IZ equal to AH; join BZ, and draw IF and HE parallel to BZ; then AE $=$ EF $=$ BF. Draw IG and HO parallel to AB; then the angle IHO $=$ HAE (Prop. 29 B. I.), and the angle AHE $=$ HIO; therefore, since AH $=$ HI, the triangles AHE and HIO (Prop. 26, B. I.) are equal, and AE $=$ HO; but HO $=$ ET (Prop. 34, B. I.), therefore AE $=$ ET. In like manner, it may be shown that EF $=$ FB.



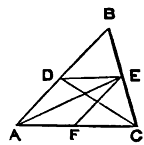
III. If the opposite sides of a quadrilateral figure be equal, it is a parallelogram. For (see figure to Prop. 34, B. 1) if ABCD be the quadrilateral, draw AB; then, in the triangles ABD, ACD, two sides of one are equal to two sides of the other, and a third side common; therefore (Prop. 8, B. 1), the triangles are equal, and the angle ADC = BAD; therefore AB is parallel to CD, and the angle BDA = CAD: therefore AC is parallel to BD.

If the opposite angles of a quadrilateral be equal, the figure is a parallelogram. For A = D, and B = C; therefore the 4 angles are double of A + B; but the 4 angles are = 4 right angles; therefore A + B = 2 right angles, and AC must be parallel to BD; and, in like manner, AB parallel to CD.

IV. A line drawn from the vertex of a triangle, bisecting the base, will bisect all lines parallel to the base, as DE. Join VD, VE, and let fall the perpendiculars DS, EP; then the triangle ADV = CEV (Prop. 38, B. 1), and ABV = VBC; hence VDB = VEB: and, since the equal triangles, VDB VEB, have a common base, VB, they have equal altitudes, and DS = EP; therefore (Prop. 26, B. 1) DO = OE.



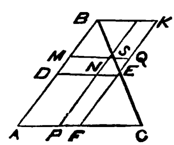
V. If the middle point of two sides of a triangle be joined, the joining line is parallel to the base, and equal to half the base. Since AD = DB, the triangle ADC is equal to DBC, and therefore equal to half the triangle ABC. In like manner AEC is equal to half the triangle; therefore, ADC is equal to AEC, and they must be between the same parallels.



In like manner, EF is parallel to AB, if F is the middle point of AC; therefore, DE = AF = half AC.

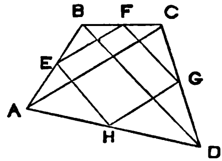
The parallelogram DF is plainly = half the triangle ABC.

VI. It is also the greatest parallelogram that can be inscribed in a triangle. For, suppose MP to be another, and draw the parallel BK, meeting PS, and FE produced, produce MS to Q, the complements DS and SK are equal; add to both the parallelogram NQ, and then DQ = NK; but, since AD = DB, FE = EK, and NK = NF;

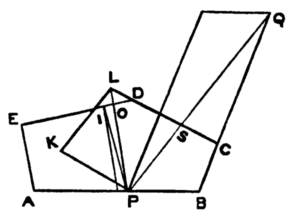


therefore DQ = NF; add to both AN, and AE = AN and DQ, that is, to AS and NQ; therefore DF is greater than AS, by the parallelogram NQ.

VII. If the middle points of the sides of a quadrilateral be joined, a parallelogram will be formed, equal to half the quadrilateral. For, EF and HG are each parallel to AC, and equal to one another, for each is equal to half AC.

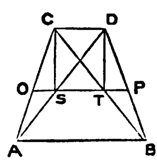


VIII. From a given point in the side of a polygon to draw a line cutting off from the polygon a given area. Let ABC DE be the given polygon, and P the given point; on PB make a parallelogram PQ, having an angle = B, and equal to twice the given area; draw its diagonal, PQ, and if it do not cut the next side LC of the polygon, PQ will obviously cut off the area required. But if PQ cut LC in S, then on PS describe a parallelogram, PL, with LSP for its angle, and = twice the triangle SCQ; draw its diagonal, PL, and if this does not cut ED, PL is the line required. But if PL cuts ED in O, on PO make a parallelogram, PI, = twice the triangle LOD, and PI will be the line required. It is plain this process may be continued until at last a line be found entirely included within the polygon, and cutting off the given area.



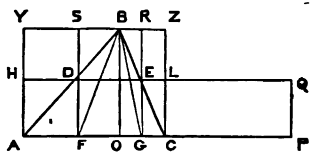
COR. A triangle or quadrilateral may thus be bisected or trisected from a point given in one of the sides.

IX. If CD and AB be parallel lines, and the lines AC, AD, BC, BD, be drawn; then, if any parallel, as OP, be drawn, OS = TP. For, if not, suppose OS > TP, then the triangle OCS will be > PDT, and also OSA > BPT; therefore ASC > BTD; add to these the equals SCD, TCD, and then ACD > BCD, which is absurd.



[N.B.—The symbol $>$ means "greater than;" and $<$ means "less than."]

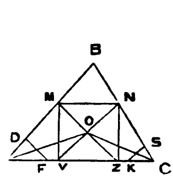
X. To inscribe a square in a triangle. This problem admits of a variety of solutions by the first book of Euclid, which are instructive in showing the different principles which may be employed in the solution of questions.



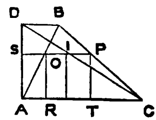
Make CP $=$ BO, altitude of the given triangle ABC, and (Prop. 45, B. 1) on AP make a rectangle, AQ, $=$ double of ABC; then DE the segment of QH, interrupted by the sides, will be the side of the square. For, draw YBZ, parallel to AC, and meeting the perpendiculars AY, FS, GR, CZ; AZ $=$ AQ, therefore (taking AL from both) HZ $=$ CQ; but (Prop. 43 B. 1) the complement HS $=$ DO, and RL $=$ EO; therefore, HZ, or HS $+$ DQ $+$ RL, $=$ DO $+$ EO $+$ DQ, or FR $=$ HZ $=$ CQ; but FS $=$ CP, therefore, as the rectangles CQ and FR are equal, and as a side of one is also equal to a side of the other, the remaining side FG $=$ CL $=$ GE, and FE is a square.

Cor. If $b =$ base, $p =$ altitude, $s =$ side of square, AP $=$ $b + p$, HQ $= s\,(b + p)$, AZ $= b\,p$; but, since AZ $=$ AQ, $s\,(b + p) = b\,p$, and $s = \dfrac{b\,p}{b + p}$, which the algebraist will perceive to be the expression for half the harmonic mean between base and altitude.

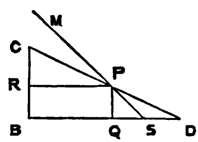
XI. Another method. Draw two lines, DE, KS, each making half a right angle with the base, AC. From A and C draw two lines, bisecting DE, KS, and meeting in O; then draw MOZ and VON, parallel to DE, KS; there will be the diagonal of the square. For, as (No. IV.) AO bisects MZ in O, and CO bisects VN in O, and since VOZ has its base angles each $=$ half a right angle, VOZ is a right angle, and also VO $=$ OZ, and VN $=$ MZ. Now, equal lines bisecting each other at right angles must be the diagonals of a square.



XII. A square may easily be inscribed in a right angled triangle, by bisecting the right angle; and then this bisector will be (as is easily shown) the diagonal of an inscribed square. From this a method is derived of inscribing a square in any given triangle, ABC. Draw BD parallel to AC, meeting the perpendicular AD in D; then in the right angled triangle, DAC, inscribe the square AI (by what precedes): OP will be the side of the square inscribed in the given triangle. For (by No. IX.) SO = IP; therefore, SI = OP, but SI = SA = OR; therefore, OR = OP.



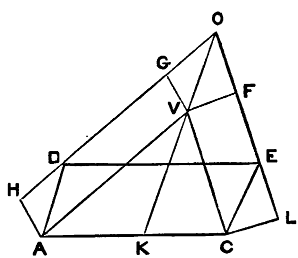
XIII. To inscribe in a given right-angled triangle a rectangle whose sides should have a given sum. Make BS = half the given sum, and draw SPM, making half a right angle, QSP; PB will be the rectangle required. For, SQ = QP, and BQ + QP = BS.



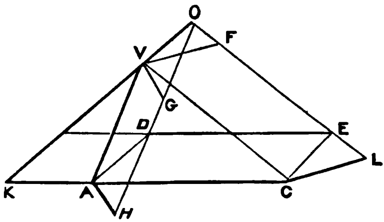
Hence, we may inscribe in any triangle a rectangle whose sides shall have a given sum, by proceeding just as in Article XII.

XIV. Euclid, I. 47, is a particular case of a theorem of Pappus, which, when enunciated in its most general form, is as follows:—

If parallelograms be described on two of the sides of a triangle, and a parallelogram also be described on the base, having a side equal and parallel to the line joining the vertex of the triangle with the point of concourse of those sides of the first made parallelograms which are parallel to the sides of the triangle; the parallelogram on the base is equal to the sum or difference of the other two, according as they are similarly or dissimilarly placed with respect to the triangle. AG, CF, are the parallelograms on the sides AE, that on the base having its side, CE, parallel to OV, the line joining the vertex with the point



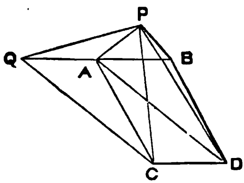
of concourse; thus, in fig. 1, AE is equal to the sum of AG and CF: in fig. 2, it is equal to their difference. For, producing OV to K, AO and AG, on same base, AV, are equal; but AO = DK, as being on same base, AD; therefore, AG = DK. In like manner, CF = EK;



but AE is equal to the sum of EK and DK, in fig. 1, and to their difference in fig. 2.—Q.E D

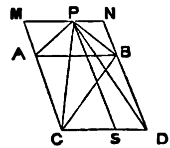
XV. The following theorem (which occurs in Mechanics) flows from the theorem of Pappus —

A triangle standing upon the diagonal of a parallelogram, and having its vertex in a given point, is equal to the sum of two triangles standing on the sides conterminous with that diagonal, and having their common vertex in the same point, if the point be external; or to their difference, if the point be internal, or APD = APB + APC.



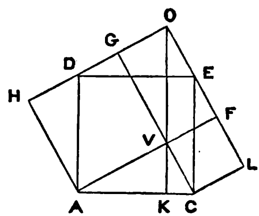
From C draw CQ, parallel to AD, meeting AB produced in Q, and join QP; the triangles APQ and APC, being halves of parallelograms standing on AQ and AC, the sides of the triangle AQC, and having P for the concourse of their sides, are together equal to the half of a parallelogram standing on QC, and having its side equal and parallel to AP; but QC is = AD, therefore the triangle APD is equal to half this latter parallelogram, and therefore equal to the sum of APC and APQ, or of APC and HPB, for APQ = APB.

Cor. Since the rectangle under the base and altitude of a triangle is equal to double the area, it follows that, if perpendiculars be let fall from any point on the diagonal and two conterminous sides, and if a and b be the sides, d the diagonal, and v'' p''' the perpendiculars, $dp''' = ap' \pm bp''$.



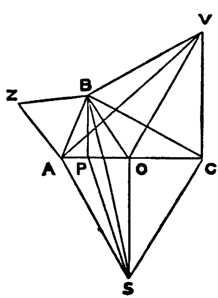
XVI. The theorem in Art. XV. may also be proved thus :—

Draw the parallel MN, meeting CA and DB, produced, and also the parallel PS; the quadrilateral, AP—BC, is equal to half the parallelogram MD (since APB = half MB, and ABC = half AD); but, since APC = $\frac{1}{2}$ AS, and BPD = $\frac{1}{2}$ BS, and APB, as before, = $\frac{1}{2}$ MB, APC + BPD + APB = $\frac{1}{2}$ MD; and therefore = quadrilateral APBC = triangles APC + BPC. Take away APC from both, and BPC remains, = APB + BPD.



XVII. The 47th Prop. of B. 1 is a particular case of the theorem of Pappus. To prove this, it is only necessary to show that the parallelogram, ADEC, described as in the theorem of Pappus, becomes a square under the conditions of the 47th Prop.

Since OF (= AH) = AV, and VF = VC, and the angle AVC = OFV, being right angles, the triangle OFV = AVC (Prop. 4, B. 1); therefore, OV = A C; also, the angle OVF = ACV, but ACV with VAC is equal to a right angle, and OVF = AVK; therefore, AVK with VAC = a right angle; therefore, VKA is a right angle; and, as CE is parallel to OV, ECA is a right angle; therefore, ADEC is a square, as its sides are equal, and its angles right angles.

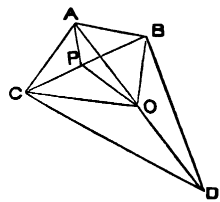


XVIII. The equilateral triangle on the hypothenuse is equal to the sum of the equilaterals on the two sides about the right angle; or ASC = ABZ + BVC.

Draw BP perpendicular to the hypothenuse, and draw AV, BS, and BO, VO, SO to the middle point of AC. Then (as in 49th Prop.) the triangle BCS may be proved equal to VCA (Prop. 4, B. 1). Now, VCA = ZVCO = VCOB = VCB + BOC; and BCS = BOC + COS + BOS = (as POS and BOS stand on same base, OS, and between same parallels, BP and OS) BOC + COS

+ POS = BOC + PCS; hence, taking BOC from the equals, VCB = PCS, and in like manner it may be shown that BAZ = APS; therefore, ACS = BAZ + BVC.

XIX. Prop. 11, B. 2. If the rectangle under AH and HB be cut off from the square AG, and also from the rectangle HD, the remainders will be equal; but the remainder of AG will be the rectangle under AH and the difference of AH and HB, and the remainder of HD will be the square of HB; hence, the rectangle under the greater segment and the difference of the segments is equal to the square of the less.



XX. The sum of the squares of the four sides of a quadrilateral is equal to the sum of the squares of its diagonals plus four times the square of the line joining the middle points of the diagonals.

If ABCD be the quadrilateral, AD and BC its diagonals, O the middle point of AD, and P the middle point of BC, join AP, PD, BO, OC, and OP; then (Prop. 13, B. 2, Cor. 1) the sum of the squares of AC and AB is equal to double the square of BP and double the square of AP, and the sum of the squares of BD and DC is equal to double the square of DP and double the square of BP; therefore, the sum of the squares of the four sides is equal to four times the square of BP, together with double the sum of the squares of AP and PD; but (by the same Cor.) the sum of the squares of AP and PD s equal to double the square of AO and double the square of PO; hence, the sum of the squares of the four sides is equal to four times the square of BP, four times the square of AO, and four times the square of PO, that is, to the squares of BC and of AD, together with four times the square of PO.

Cor. 3. If the quadrilateral is a parallelogram, the diagonals bisect one another, and the line PO vanishes; therefore, the sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of its diagonals.

XXI.—Compound Ratio.

(*Book 5, Def.* 12.)

1. If any number of ratios be such that the consequent of each

is the antecedent of the next, the first antecedent is said to have to the last consequent a ratio compounded of them all. If a, b, c, d, e be any magnitudes, then $a : e$, in a ratio compounded of $a : b$, $b : c$, $c : d$, and $d : e$; and the notation by which this is expressed is,

$$a : e :: \left\{\begin{array}{c} a : b \\ b : c \\ c : d \\ d : e \end{array}\right\}$$

2. Any ratio $a : b$, may be resolved into any number of ratios by observing the same condition of having the consequent of each the antecedent of the following. Take any magnitudes, p, z, e, m, and

$$a : b :: \left\{\begin{array}{c} a : p \\ p : z \\ z : e \\ e : m \\ m : b \end{array}\right\}$$

For example, $2 : 3 :: \left\{\begin{array}{c} 2 : 7 \\ 7 : 19 \\ 19 : 5 \\ 5 : 3 \end{array}\right\}$

3. Prop. 23, B. 6, gives us the means of compounding ratios, which do not answer these conditions. Suppose it be required to compound $a : b$ and $m : n$. By Prop. 1, B. 6, $a : b :: am : bm$, and $m : n :: bm : bn$. Now, compounding the ratios, we have $\left\{\begin{array}{c} am : bm \\ bm : bn \end{array}\right\} :: am : bn$; whence it appears that ratios are compounded by multiplying the antecedents and consequents.

4. If $a : b :: b : c$, $a : c :: \left\{\begin{array}{c} a : b \\ b : c \end{array}\right\} :: \left\{\begin{array}{c} a : b \\ a : b \end{array}\right\}$; hence, duplicate ratio is a ratio compounded of two equal ratios.

Also, $a : c :: a^2 : b^2$, the ratio of the squares.

In like manner, triplicate ratio is compounded of three equal ratios, and is the ratio of the cubes.

5. If $a : b :: b : c$, then a ratio compounded of $\left\{\begin{array}{c} a : c \\ a : b \end{array}\right\}$ is said to be the sesquiplicate of $a : c$.

The sesquiplicate ratio is, therefore, a ratio compounded of the given ratio $a : c$ and its own subduplicate $a : b$.

6. The algebraist will perceive that if ratios be considered as fractions, to compound ratios is the same thing as to multiply

fractions; antecedents and consequents being equivalent to numerators and denominators; as

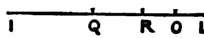
$$\frac{a}{c}=\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}\times\frac{d}{e};\ \text{also,}\ \frac{a}{b}\times\frac{m}{n}=\frac{am}{bn}.$$

$$\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2};\ \frac{a}{b}\times\frac{a}{b}\times_b=\frac{a^3}{b^3};\ \frac{a}{b}\times\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}=\frac{a^{\frac{3}{2}}}{b^{\frac{3}{2}}};$$

The three last examples corresponding to duplicate, triplicate, and sesquiplicate ratios.

XXII.—On the Sum of a Decreasing Geometric Series.

If the given ratio be a ratio of less inequality, LO : LR, it may be continued in an increasing series, until a magnitude be found greater than any assigned. For, let LO, LR, LQ, LI be in continued proportion; then LO : LR :: LR : LQ; therefore, convert°. LO : OR :: LR : RQ;



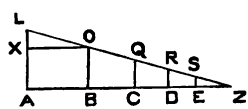
but LR is > LO; therefore, RQ > OR (Prop. 23, B. 5); in the same manner, IQ > RQ; and, therefore, since the magnitudes added to the first are in an increasing series, a magnitude can be found greater than any assigned.

If the given ratio be a ratio of greater inequality, AB : CB (or the series a decreasing one), the series can

be continued until a magnitude be found less than any assignable. Let the assigned quantity be OL; make OL : OR :: BC : BA, and continue the series until IL greater than AB be arrived at; then continue the ratio AB : CB to the same number of series, of which let the last be FB; then FB is less than OL. For, there are two series of magnitudes, proportional and equal in number, and it follows, *ex æquo perturbatè*, that AB : FB :: IL : OL; therefore, FB > OL (Prop. 23, B. 5).

If there be a series of magnitudes, decreasing in continued proportion, and this series be conceived to be continued *ad infinitum*, the sum of such an infinite series can be found.

1. The ratio AB : BC may be continued in a series by the following construction. Draw AL perpendicular to and equal to AB, and BO perpendicular to and equal



to BC; join LO, and produce to cut AC, produced in Z; then raise the perpendicular CQ, and take CD = CQ; raise the per-

pendicular DR, and take DE $=$ DR; the lines AL, BO, CQ, DR, &c., are in continued proportion. For AZ : BZ ($::$ AL : BO) $::$ AB : BC, altern° AZ : AB $::$ BZ : BC; convert° AZ : BZ $::$ BZ : CZ, but AZ : BZ $::$ AL : BO, and BZ : CZ $::$ BO : CQ; therefore, AL : BO $::$ BO : CQ; and in like manner it may be proved that the other perpendiculars are in continued proportion.

2. Since AL : BO $::$ AZ : BZ, and since AL $<$ AZ, BO is $<$ BZ, and CQ, DR, &c., are less than CZ, BZ, EZ, &c.; but since AL $+$ BQ $=$ AC, AL $+$ BO $+$ CQ $=$ AD, AL $+$ BO $+$ CQ $+$ DR $=$ AE, and so on: or generally, the sum of all the perpendiculars but the last is equal to the distance between the first and last; and, since the last is less than the remaining portion of AZ, it follows that the sum of all the perpendiculars, howsoever far continued, shall not be greater than AZ; and, since the sum of the perpendiculars falls short of AZ by the quantities CZ, DZ, EZ, &c., successively, which are in continued decreasing proportion, it follows that the series of perpendiculars may be continued until it shall fall short of AZ by a quantity less than any assignable. Therefore, AZ is a magnitude which the sum of the series, if continued without limit, cannot exceed, and from which, nevertheless, this sum differs by a magnitude less than any assignable. AZ is, properly speaking, the limit of the sum of the series.

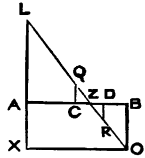
3. Since LX : XO $::$ AL : AZ, and since LX is the difference between the first two terms, and XO $=$ first term, it follows that the sum of the series : first term $::$ first term : difference of first and second.

Let a and b be the two first terms, and also let $a : b :: 1 : r$, so that $b = ar$; then, if s be the sum (AZ) $s = \dfrac{a^2}{a-b}$, or $s = \dfrac{a^2}{a-ar} = \dfrac{a}{1-r}$; for example, if $a : b :: 2 : 1$, $r = \frac{1}{2}$, and $s = \dfrac{a}{1-\frac{1}{2}} = 2a$; again, if $a : b :: 1 : \frac{1}{3}$, $r = \frac{1}{3}$ and $s = \dfrac{3a}{2}$.

4. The rectangle under the sum and the second term is equal to the rectangle under the first term and the difference between it and the sum, or $sb = a\ (s - a)$. For (AL $=$) AB : AZ $::$ (BO $=$) BC : BZ; therefore AB $\times$ BZ $=$ AZ $\times$ BC.

Hence, if the sum, AZ, and second term, BC, be given, the first may be found by cutting AZ into two such parts that their rectangle AB $\times$ BZ may be equal to the given rectangle AZ $\times$ BC.

5. If there be a decreasing series, as before, and if the terms be alternately subtracted from and added to the first, the sum may be found. Draw AL and BO perpendicular to AB, but in opposite directions; join LO, cutting AB in Z; assume BC $=$ BO, and raise CQ; assume CD $=$ CQ, and raise DR, and so on; then AL, BO, CQ, DR, &c., are continued proportionals, and AZ is the sum of AL $-$ BO $+$ CQ $-$ DR $+$, &c., continued *ad infinitum*. For, AZ : BZ $::$ (AL : BO) $::$ AB : BC; therefore, altern°. AZ : AB $::$ BZ : BC; convert°. AZ : (AB $-$ AZ $=$) BZ $::$ BZ : (BC $-$ BZ $=$) CZ; and hence, as in the former proof, AL, BO, CQ, &c., are in continued proportion, being proportional to AZ, BC, CZ, &c.

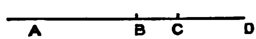


Now, AL $-$ BO $=$ AC, AL $-$ BO $+$ CQ $=$ AD, AL $-$ BO $+$ CQ $-$ DR $=$ AE, and these successive sums differ from AZ by quantities CZ, DZ, EZ, decreasing in continued proportion; whence, as before, AZ is the limit of the sum of the series.

In this case r is algebraically called negative, and $s = \frac{a}{1+r}$

XXIII.—Harmonic Proportion.

1. A line is said to be cut harmonically when, being divided into three parts, the rectangle under the whole line and the middle segment is equal to the rectangle under the extreme segments, or AD $\times$ BC $=$ AB $\times$ CD; or the line is cut harmonically when AB : BC $::$ AD : DC.



2. AB, AC, AD are said to be in harmonic proportion; likewise (measuring from the other end) DC, DB, DA.

AC is the harmonic mean between AB and AD; DB the harmonic mean between DC and DA.

3. Of three harmonic proportionals, 1st : 3rd $::$ difference of 1st and 2nd : difference of 2nd and 3rd; for AB : AD $::$ BC : CD.

4. The rectangle under the extremes is equal to the square of the mean, together with the rectangle under the differences. For,

$$\begin{aligned} DA \cdot AB &= AB^2 + AB \cdot BC + AB.\,CD. \\ &= AB^2 + AB \cdot BC + AD.\,BC. \\ &= AB^2 + AB \cdot BC + BC^2 + AB \cdot BC + BC \cdot CD. \\ &= AC^2 + BC \cdot CD. \end{aligned}$$

5. If the mean (AC) be bisected in O, OB : OC ∷ OC : OD. For, AB : BC ∷ AD : DC; hence, by division and composition, AB — BC : AB + BC ∷ AD — DC : AD + DC, or 2 OB : 2 OC ∷ 2 OC : 2 OD.

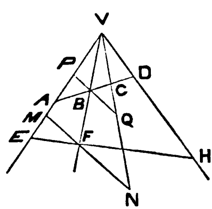
6. In the same figure, the whole line, AD, and the part, OC, are similarly cut in B, or AB : BD ∷ OB : BC. For, by conversion from No. 5, OB : BC ∷ OC : CD ∷ AO : CD; hence, AO : OB ∷ DC : CB, or comp°. AB : OB ∷ DB : CB, or altern°. AB : DB ∷ OB : CB.

7. The rectangle under the mean and the sum of the extremes is equal to twice the rectangle under the extremes. By 6, AB : BD ∷ OB : BC; but, from No. 5, convert°. OB : BC ∷ OC : CD; therefore, AB : BD ∷ OC : CD; compon°. AD : DB ∷ OD : DC; whence, AD × DC = DB × OD; 2 OD = AD + DC, and DB is the harmonic mean between AD and AC.

8. Hence, if a and b be the extremes, the mean $= \dfrac{2\,ab}{a + b}$ or to $\dfrac{ab}{\frac{1}{2}\,(a + b)}$.

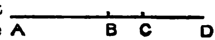
9. Since $ab =$ square of geometric mean, and $\frac{1}{2}\,(a + b) =$ arithmetic mean, it follows that the geometric mean is itself a mean between the harmonic and arithmetic means.

10. If four lines, diverging from a point, V, cut any line, ABCD, harmonically, any other line, EFGH, will be cut also harmonically. For, draw BPQ parallel to VD, then DA : AB ∷ DV : BP, and DC : CB ∷ DV : BQ; therefore, DV : BP ∷ DV : BQ; therefore, BP = BQ; therefore, if MFN be drawn parallel to PQ, MF = FN; whence VH : MF ∷ VH : FN; but VH : MF ∷ HE : EF, and VH : FN ∷ HG : to GF; therefore, HE : EF ∷ HG : GF.



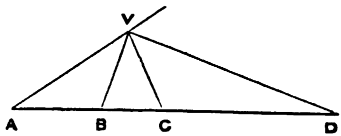
The four diverging lines are called an harmonic pencil.

11. If four points be taken in a line, the ratio of the rectangle under the whole line and middle segment to the rectangle under the extreme parts, or AD × BC : AB × CD, is called the anharmonic ratio of that system of points.

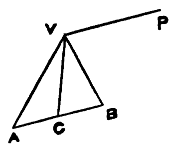


Now, $AD \times BC : AB \times CD :: \begin{Bmatrix} BC : CD \\ AD : AB \end{Bmatrix} :: \begin{Bmatrix} BQ : VD \\ VD : BP \end{Bmatrix}$ $:: BQ : BP$. Hence the ratio of the segments into which the parallel PQ, to the 4th line, intercepted between 1st and 3rd. is cut by the 2nd, is the anharmonic ratio of the pencil of four lines.

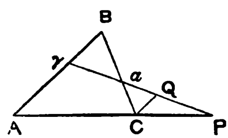
12. The bisectors of the vertical angle of a triangle, and its supplement, form with the sides an harmonic pencil. For, $AB : BC :: AD : DC$; therefore, the line AD is cut harmonically.



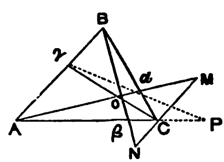
13. If, from the vertex of a triangle, a line be drawn to bisect the base, and another line drawn parallel to the base, the four lines, VA, VC, VB, VP, form an harmonic pencil. This is evident from No. 10, as the parallel AB is bisected in C.



14. If a line, YXP, be drawn, cutting the three sides of a triangle, the segments of each side are in a ratio compounded of the ratios of the segments of the other sides. Draw CQ parallel to AB, then $AP : PC :: A\gamma : QC :: \begin{Bmatrix} A\gamma : \gamma B \\ \gamma B : CQ \end{Bmatrix} :: \begin{Bmatrix} A\gamma : \gamma B \\ B\alpha : \alpha C \end{Bmatrix}$ Hence, $AP \cdot B\gamma \cdot C\alpha = \alpha B \cdot PC \cdot \gamma A$.

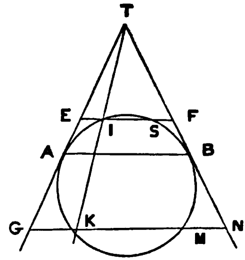


15. If lines be drawn from the angles through any point, O, the segments of any side are in a ratio compounded of the ratios of the segments of the other two. Draw MCN parallel to AB; then $A\beta : \beta C :: AB$ $CN :: \begin{Bmatrix} AB : CM \\ CM : CN \end{Bmatrix} :: \begin{Bmatrix} B\alpha : \alpha C \\ A\gamma : \gamma B \end{Bmatrix}$ Hence, $A\beta \cdot B\gamma \cdot C\alpha = \alpha B \cdot \beta C \cdot \gamma A \cdot$ as before.



16. If $\alpha\gamma$ be joined, and produced to P, $AP \quad PC :: A\beta$ βC, and the line AP is cut harmonically.

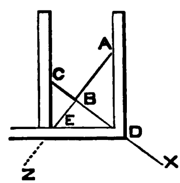
17. If tangents be drawn from T, any secant, as TK, is cut harmonically by the circle, and the chord of contact (AB). For, draw the parallels EIF, GKM, meeting TA in E and G; then as the line from T to the centre bisects both IS and EF, EI $=$ SF, and GK $=$ MN; now, as $EI : IF :: GK : KN$, it follows that $EI \cdot IF : GK \cdot KN :: EI^2 : GK^2$ (Prop. 20, B. 6, as they are similar rectangles); but $AE^2 = EI \cdot IF$ and $GA^2 = GK \cdot KN$; therefore, $GA^2 : AE^2 :: GK^2 : EI^2 :: GT^2 : TE^2$; hence, $GA : AE :: GT : TE$, and GT is cut harmonically; therefore, also, TK is cut similarly.



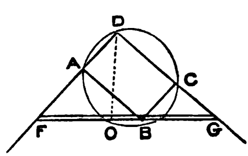
XXIV.—On the Methods of Finding two Mean Proportionals.

(*Book* 6, *Proposition* VIII.)

1. Plato's method of finding two mean proportionals:—Construct a frame so that a ruler may be capable of sliding along one side of a right angle, and may be always perpendicular to it. Let the given extremes, AB and CB, be placed at right angles, and produced indefinitely; then apply the angle D to BX, so that when the fixed side passes through A, and the sliding perpendicular through C, the produced part of the other extreme may pass through the foot of the ruler, E. Then BE and BD are the two means; for (Prop. 8, B. 6, Cor.) BD is a mean between AB and BE, and BE is a mean between CB and BD.

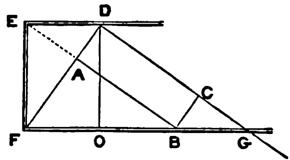


2. *Philo's Method.*—Place the extremes, AB and BC, at right angles; complete the rectangle, AC; circumscribe it with a circle, and apply a ruler to pass through B, so that, if the sides of the rectangle be produced, BG may $=$ OF. For, then $BF \times FO = OG \times GB$; and therefore, $DF \times AF = DG \times GC$; and there-



fore, AF : GC :: DG : DF; but DG : DF :: BA : AE, and DG : DF :: GC : CB; therefore, BA, AF, GC, and CB are in continued proportion.

3. It can be shown that these two methods are substantially the same. For, as in Philo's method, DO is perpendicular to FG (DB being a diameter), the problem is reduced to drawing a line through a point B, given between the legs of a right angle, FDG, so that a perpendicular being let fall from D, the segment OF may be equal to BG.

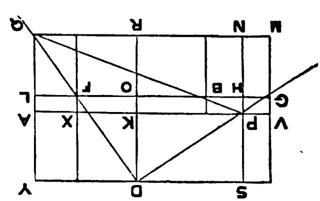


Now, supposing this to be done, apply the side of Plato's square to FG, and move the sliding ruler to pass through D; then shall BA produced, pass through E. For, FO = DE, being opposite sides of a rectangle; therefore, BG = DE, and the lines joining their extremities are equal and parallel; therefore, as AB is already parallel to DG, AE and AB form one right line.

4. Again, it may be shown that it is *possible* to apply the ruler so as to fulfil the conditions of Philo's method. For, as the ruler rotates on the point B, the segment OF may be made to pass through every degree of magnitude, from cypher, when F coincides with A, to infinity, when BF becomes parallel to DA; and during these changes BG increases as OF diminishes, and *vice versa*. Also, the segment BG may be made to pass through every degree of magnitude, from BC, when G coincides with C, to infinity, when BG becomes parallel to DC; and during these changes OF increases as OF diminishes; hence, as OF may be made greater and less than BG, a position may be found when they are equal.

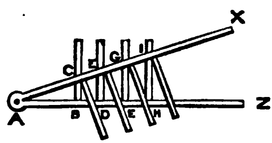
5. It follows, from the identity of the methods of Plato and Philo, that Plato's method is also possible.

6. The line drawn as in Philo's may be shown to be the shortest line that could be drawn through B, and terminated by the sides of the right angle; or, more generally, if FDG be any angle, the line through a given point B is a minimum when BG



= FO, the segment cut off by a perpendicular from D. For, draw any other line, PBQ; then draw parallels to FG through D, P, Q, and perpendiculars to these parallels through G, P, B, D, F, and Q. Then FY = FR (Prop. 43, B. 1) = BM (as OF = BG); also, BN = (being complements) BA, and VS = AK; therefore, GS = GK = BX (as GO = BF). Now, BM > BN or BA, and BA > BX or GS; therefore, BM or FY > GS; therefore, FL > GH, and HL > GF; but PQ > PA or HL; therefore PQ is much greater than GF.

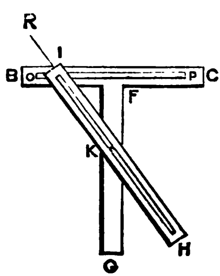
6. *Descartes' method of finding two or more mean proportionals.*—AX, AZ are two rulers, hinged at A, and each furnished with a groove, in which are inserted two series of rulers, alternating with each other, as in the figure, and capable of sliding in the groove. Then, taking AB = the lesser extreme, let the first ruler, BC, be made fast by a screw at B; and then, as AX, AZ, are opened, the ruler BC will push forward CD, and CD push forward DE, and so on.



If two means are required, open the rulers until AE = greater extreme; then AG, AD will be the means (by Prop. 8, B. 6).

If three means are required, open them until AF = greater extreme, and then AG, AD, AE will be the means, and so on.

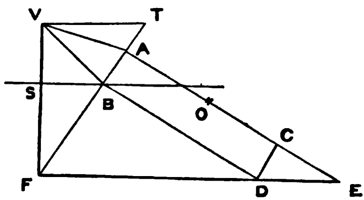
7. The trammel of Nicomedes was an instrument invented by that geometer to effect mechanically the problem on which the finding of two mean proportionals may be shown to depend, namely, "to draw a line through a given point, so that the segment intercepted by the legs of an angle given in position may be of a given length, M."



The trammel consists of a frame shaped like the letter T, which has a groove, OP, in the upper part, BC, and a fixed pin at K. On this a flat ruler, IH, with a groove, is so adjusted that a pin at one extremity slides in the groove OP, while the moveable ruler, IH, slides backwards and forwards over the pin K. Then, a stem being attached, so as to coin-

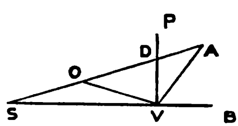
cide with IH, produced, let PO coincide with one leg of the given angle, and suppose N to be the length of the perpendicular on that leg drawn from the given point; then assume a point R, on the stem, such that IR : KF $::$ M : N, and move the ruler IH, so that R may fall somewhere on the other leg of the angle; then the line RI will correspond in position to the line required, and a figure may be constructed similar to that found by the trammel, and having its lines : corresponding lines in the latter $::$ N : KF.

8. The two problems, of finding two mean proportionals, and of the trisection of an angle, may both be reduced to the problem, "to insert a line of given length between the legs of a given angle, so that the line may also pass through a given point."

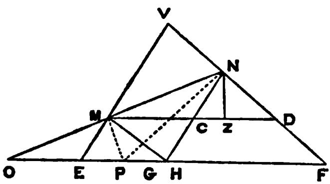


This may be shown in the first problem by the following analysis. It is plain from the proofs in Philo's method, that if AB and AC be the extremes, and if AF. FB can be made = AE. EC, BF and CE shall be the means. Bisect AC in O, and (Prop. 6, B. 2) $OE^2 = OC^2 + AE.\ CE = OC^2 + AF.\ FB$; make BVA an isosceles triangle, having each side = OC, and then $OE^2 = BV^2 + AF.\ FB = VF^2$, and OE = VF; divide VF, so that VS = CE, and SF = OC; then, as BF. CE = (by similar triangles) AB. AC, BF. VS = AB. AC = 2 AB. AO = 2 AB· SF, or (taking AT = AB) = BT. SF; therefore, BF : BT $::$ SF : SV, and VT is parallel to BS; but the points V and T are given, and therefore the parallel BS is given, and the angle SBF given in position; hence, the problem can be solved, if from V the line VF can be drawn, so that SF = VB ½ AC, a given line.

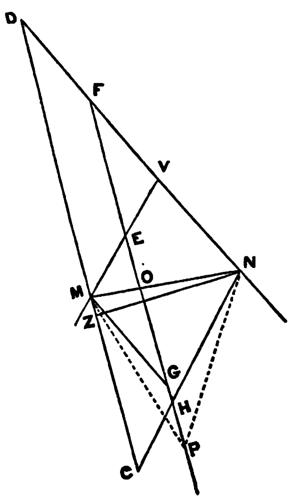
9. Let AVB be an angle to be trisected. Raise VP perpendicular to VB; then, if AS can be drawn so that SD = 2 AV, the angle ASV shall be ⅓ of AVB; for, draw VO to the middle point of SD, and therefore = ½ SD; then DOV = 2 OSV but OV =



VA, and therefore DOV $=$ VAO, hence AVB $=$ OSV $+$ VAO $=$ OSV $+$ 2 OSV $=$ 3 OSV.



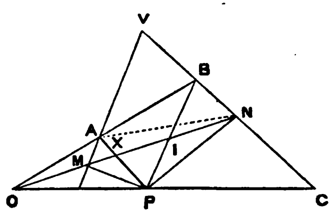
XXV. Problems relating to the investigation of maxima and minima are of frequent occurrence, and a sample is here given of one which is both instructive in its details and fruitful in its results. "Given, an angle in position and two points, to draw a line through one of them, so thatthe triangle standing on the segment of this line intercepted by the legs of the angle, and having its vertex in the other point, may be a maximum or a minimum, or of a given area."



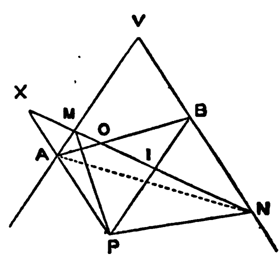
Let MVN be the angle, O and P the given points (which in fig. 1 are at different sides of a leg—in fig. 2 between the legs), the triangle MNP is to be of a given area, from which we can determine the limits of its magnitude. Proceeding analytically, let MG, NH, MD be drawn parallel to VF, VE, EF, and NZ perpendicular to CD. The triangle MNP $=$ ONP $\mp$ OMP; therefore, since OP is their given common base, the difference or sum, NZ, of their altitudes is given; but NZ : CD :: altitude : base of VEF, which is given, and therefore CD is given. Now, CD $=$ MD $\mp$ MC $=$ FG $\mp$ EH $=$ FO $\mp$ OG $\mp$ (OH $\mp$ OE) $=$ OF $+$ OE $\mp$ (OG $+$ OH); but since OF and OE are given

OG + OH will be given, and CD will be a maximum (in fig. 1) when OG + OH is a minimum, and (in fig. 2) CD will be a minimum when OG + OH is a maximum. But since OE : OH (∵ OM : ON) ∷ OG : OF, OG × OH = OE × OF, which is given. Hence, the sum of OG and OH and their rectangle is given, and the lines themselves can be found, and therefore the points G and H, and thence the points M and N. Further, since OG × OF is given, OG + OF is a minimum, therefore OG and OF are equal, and the parallels MG, NH meet in a point such that its distance from O is a mean proportional between OE and OF.

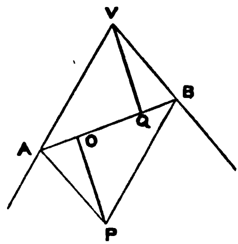
XXVI. The problem may be solved also on more elementary principles. It may be proved that when AB is so drawn as that it shall pass through O, and that PA and PB shall be parallel to the sides of the angle, the triangle APB is a maximum in fig. 1, and a minimum in fig. 2.



For, let MNP be any other triangle; then, as PA and BN are parallel, the triangle PAB = PAN; but, as ANM > PAM (by reason of PB and MV being parallel), it follows (by subtracting and adding the common portions) that PAN > PMN, in fig. 1, and PAN < PMN, in fig. 2; therefore, PAB is greater than PMN in the first case, and less than it in the second case.

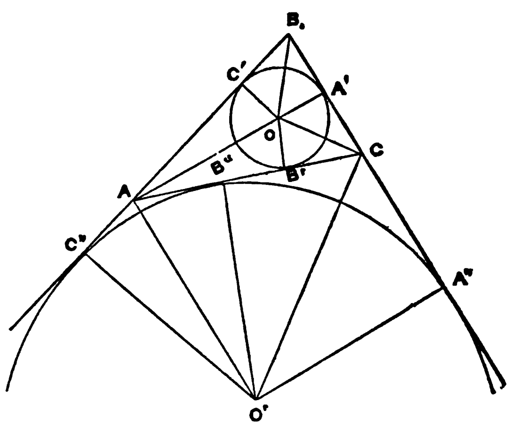


It may, then, be shown easily (by Prop. 1, B. 6) that even when P is not at the point of concourse of the parallels AP, BP, or at any other point of the line joining OP, the conditions of the maximum or minimum are the same.



XXVII.* The line drawn as in Philo's method, may hence easily be shown to be a minimum, *i.e.*, when the perpendicular, QV, cuts off BQ = AO. For, complete the parallelogram AVBP, and join PO; (by Prop. 4, B. 1) triangle PAO = VQB, and therefore PO is perpendidular to AB. Now, by the precedingarticle APB is less than any other triangle having its vertex at P, and its base passing through O, and terminated by the legs of the angle; but PO is greater than a perpendicular drawn from P to any other line drawn through O (Prop. 19, B. 1); since, therefore, the area is less and the altitude greater, the base AB must be less than any other such base.

XXVIII. Given, in numbers, the three sides of a triangle, to find its area. The following elegant theorem may be proved geometrically:—If s be $= \frac{a+b+c}{2}$, which is called the semi-perimeter, then the area is = to $\sqrt{s(s-a)(s-b)(s-c)}$.

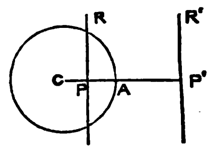


Let a circle be inscribed in the triangle, and another touching the

base and two sides produced (the centre of which is found in the point where the bisectors of the angles below the base intersect); then, as C″, B″ are points of contact, the tangent AC″ = AB″, and CB″ = CA″; therefore BC″ + BA″ = sum of three sides; and as BC″ = BA″, being tangents, BC″ = s. Again, since BC′ = BA′, AC′ = AB′, and CA′ = CB′, it follows that 2 BC′ + 2 AC′ + 2 CA′ = perimeter = 2 BC″; therefore, BC′ + AC′ + CA′ = BC″ = BC′ + AC′ + AC″; therefore, AC″ = CA′ = CB′; and AC′ = CB″; hence, C′C″ = AC, and BC′ = $s - b$; also AC″ = BC″ — AB = $s - c$; and CA″ or C′A = BA″ — BC = $s - a$. Also, since the triangles BOA, BOC, AOC, have equal altitudes, namely, the radius of inscribed circle (r), the area of ABC is $= \frac{r(a+b+c)}{2} = rs$.

Now, BC′ : OC′ :: BC″ : O′C″, or $s - b : r :: s : r'$ (calling O′C″ $= r'$) therefore, $s\,(s-b) : rs :: rs : rr'$; but, by similar triangles, AO′C″ and AOC′, rr' = AC″ × AC′ = $(s-a)(s-c)$; hence, $s\,(s-b)$: area :: area : $(s-a)(s-c)$, and area $= \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$.

XXIX. If two points on a diameter, and on the same side of the centre of a circle, be so placed that the rectangle under their distances from the centre be equal to the square of the radius, they possess several remarkable properties. If perpendiculars to the diameter be drawn at these points, each point is called *a pole*, and the perpendicular through the other is called *the polar line*, or simply *the polar*. Thus, P and P′ being points such that CP × CP′ = CA²; P′R′ is the polar of the pole P, and PR the polar of the pole P′.



Some of the properties of polars may here be mentioned:—

a. Any line drawn through the pole, cutting the circle and polar, is cut harmonically.

b. If any point on the polar be taken as a new pole, then its polar will pass through the first pole.

c. If any number of lines pass through a point, the locus of the poles of these lines will be the polar of the point.

d. If tangents be drawn at the points where lines through a given point cut a circle, the locus of the intersection of these tangents will be the polar of the point.

e. If any secants be drawn through a given point, the locus of the intersection of the lines joining the points where the secants cut the circle, will be the polar of the given point.

XXX. *Of loci.*—When the conditions given for finding a point are such as not to fix its position, but only to determine that it must be found somewhere in a given line, or circle, or other curve, such line or curve is called *the locus* of the point. For example, when the base and area of a triangle are given, its vertex must be found somewhere in a parallel to the base, and this parallel is called the locus of the vertex: when the base and vertical angle are given, the locus of the vertex is the segment of a circle (Prop. 21, B. 3): when the base and sum of squares of sides are given, the locus is a circle having its centre in the middle point of the base (Cor. Prop. 13, B. 2).

The determination of loci under given conditions belongs to an interesting and important class of problems.

THE END.

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